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ON DIFFUSION BY DISCONTINUOUS MOVEMENTS, AND ON THE TELEGRAPH EQUATION

By S. GOLDSTEIN
(*Institute of Technology, Haifa, Israel*)

[Received 8 August 1950]

SUMMARY

At time $t = 0$ a large number of non-interacting particles start from an origin and move with a uniform velocity v along a straight line for an interval of time τ. To begin with, half move in each direction. Thereafter, and at the end of each successive interval of time τ, each particle starts a new partial path; it still moves with speed v, and there is a probability p that it will continue to move in the same direction as in its previous path, and a probability q $(= 1-p)$ that the direction of its velocity will be reversed, so the directions in any two consecutive intervals are correlated with a correlation coefficient $c = p-q$. The partial correlations for non-consecutive intervals are zero. The difference equation is found for the fraction $\gamma(n,\nu)$ of the number of particles at a distance $y = \nu v\tau$ from the origin after a time $t = n\tau$; it is shown how $\gamma(n,\nu)$ may be computed; asymptotic formulae for large n are found, both for a fixed value of p/q and for a fixed value of nq/p. The limiting density distribution (and the limiting characteristic function) are found when $n \to \infty$, $\tau \to 0$, with $n\tau = t$, $\nu v\tau = y$, and in the limiting operation $c = 1-\tau/A$, with A constant, so that $c \to 1$, and the speed v is kept constant; the limiting form of the difference equation is the telegraph equation, with $v^2 = (LC)^{-1}$, $A = L/R$, where L, C, and R are the self-inductance, capacitance, and resistance per unit length; and the limiting density distribution is the solution of this equation for an instantaneous source. If $p+q \neq 1$, and there is, at the end of each interval of time τ, a non-zero probability, $1-p-q$, that a particle will escape from the system, then the same limiting operation, with $\tau/(1-p-q) = G$, G constant, leads to the telegraph equation with leakage, the leakage resistance being G/C. The solution of the telegraph equation is further considered (and in particular is given in terms of Lommel's function of two variables) when one end of a long cable is held at a constant potential for $t \geqslant 0$.

1. Introduction

BEFORE considering diffusion by continuous movements, G. I. Taylor (1) in a paper published in 1922, considered briefly an extension of the usual treatment of discontinuous random migration in one dimension. The problem is phrased as a physical problem in diffusion by considering that at time $t = 0$ a large number of non-interacting particles start from an origin and move with a uniform velocity v along a straight line for an interval of time τ, during which each moves a distance $d = v\tau$; to begin with, half move to the right (in the positive direction along the line) and half to the left (in the negative direction); after time τ, and after each interval of time τ thereafter, each particle starts a new partial path, and again moves with velocity v through a distance d in time τ, but may either continue in the same direction as in the previous path or may move in

the opposite direction. The process continues for a large number of partial paths and intervals τ. In the usual theory of random flights there is no correlation between the directions of motion of a particle in any two time-intervals. If, however, the direction in any one interval $[m\tau, (m+1)\tau]$ is correlated with that in the next interval $[(m+1)\tau, (m+2)\tau]$ with a correlation coefficient c, and the partial correlations for non-consecutive intervals are all zero (so that the correlation coefficient between the directions in the intervals $[m\tau, (m+1)\tau]$ and $[(m+s)\tau, (m+s+1)\tau]$ is c^s), then Taylor† showed that the mean square distance from the origin after time $n\tau = T$ is

$$v^2\left\{\frac{1+c}{1-c}\tau T - \frac{2c(1-c^n)}{(1-c)^2}\tau^2\right\}. \tag{1}$$

As $\tau \to 0$ and $n \to \infty$, with v fixed, the diffusion approximates to a continuous diffusion of some kind; the mean square distance stays finite if $\tau/(1-c) \to$ a non-zero limit. If $\tau/(1-c) = A$, then in the limit the correlation coefficient between the directions of the velocity of a particle at two times separated by an interval t is $e^{-t/A}$, and the mean square distance from the origin after a time T is

$$v^2\{2AT - 2A^2(1-e^{-T/A})\}, \tag{2}$$

which becomes v^2T^2 for small T, as it should, the correlation not having fallen from unity.

These are Taylor's results. Taylor went on to consider diffusion by continuous movements; the process considered above cannot lead to turbulent diffusion. Some years ago (1938/9) I considered more fully the density distribution of the particles after n steps, and the approach to the limit. This work is reported here. (It appears from my notes that at that time Mr. J. Wishart had obtained independently the formula (16) below for the moment-generating function, and had drawn my attention to a paper (**2**) he had published on a similar problem.)

After the difference equation for the fraction, $\gamma(n, \nu)$, of the number of particles at a distance νd from the origin after a time $n\tau$, and the formula for the moment-generating function, have been found in section 3, the computation of $\gamma(n, \nu)$ is considered in sections 4, 5, 6, where asymptotic expressions are found for large n, both for a fixed value of c (which may be positive or negative) and for a fixed positive value of n/r, where $r = (1+c)/(1-c)$. The passage to a continuous distribution, when $n \to \infty$, $\tau \to 0$, $d \to 0$, $c \to 1$, with v constant, $n\tau = t$, $\nu d = y$, $\tau/(1-c) = A$, $n/r \to B = t/2A$, A constant, is considered in sections 7 and 8. The limiting density distribution, and the limiting characteristic function, and their

† c^2 occurs in place of c in the numerator of the second term in ref. **1**. It appears that on the last line of p. 199 of ref. **1**, n should be replaced by $n-1$.

connexion, are considered in section 7; in section 8 it is shown that the limiting density distribution satisfies the telegraph equation (without leakage), and with $v^2 = (LC)^{-1}$, $A = L/R$, where L, C, and R are the self-inductance, capacitance, and resistance per unit length; this equation is the limiting form of the difference equation for $\gamma(n, \nu)$; and the limiting density distribution is the solution of this equation for an instantaneous plane source at the origin. The conditions under which the limiting operation is carried out should be stressed: the partial correlations for non-consecutive intervals are all zero, $c \to 1$ in a fixed manner ($c = 1-\tau/A$), and the particle speed is constant. (With three possible velocities, for example, $\pm v$ and 0, a third-order equation is obtained (section 10).) In section 9 it is shown how a solution of the telegraph equation when one end of a long cable is suddenly raised to, and thereafter held at, a constant potential is related to the solution for an instantaneous source, and how various forms of the solution of that problem are related among themselves, with certain mathematical relations involving Bessel functions and Lommel functions of two variables. In section 10 some further developments are mentioned; in particular, it is shown that if the sum, $p+q$, of the probability p that a particle will continue to move in the same direction and the probability q that it will reverse its direction of motion, is not equal to unity, but there is a non-zero probability, $1-p-q$, that it will escape from the system altogether, then the limiting form of the difference equation when $n \to \infty$, $\tau \to 0$, with $n\tau = t$, $\tau/(1-c) = A$, $\tau/(1-p-q) = G$, is the telegraph equation with leakage, the leakage resistance being G/C; the density distribution function is easily found for this case. The solution for constant potential at one end again involves Lommel's function of two variables.

2. Notation

a_1, a_2	for definition, see equation (18).
b_1, b_2	for definition, see equation (48).
b_3	for definition, see equation (52).
$c\ (= p-q)$	the correlation coefficient between the directions of the velocities in two consecutive partial paths.
$d\ (= v\tau)$	the length of each partial path.
f	$f(u) = \int_{y=-\infty}^{\infty} e^{iuy}\, dF = M(n, ud)$.
f^*	the limit of f for the 'continuous' distribution (equation (94)).
k_1, k_2	for definition, see equation (19).
l_1, l_2	for definition, see equation (47).
l_3	for definition, see equation (52).

m	an integer.
n	the number of partial paths.
p	the probability that a particle continues to move in the same direction at the end of a partial path: the Heaviside operator in section 9.
q	the probability that the direction of motion of a particle is reversed at the end of a partial path.
$r = p/q$	
s	an integer.
t	time; used in place of T for $n\tau$ from section 8 onwards.
u	a real variable.
$u_1 = \sin^{-1}(1/r)$.	
v	the uniform speed in each particle path.
$w = (nr)^{\frac{1}{2}}u$.	
$y\ (= \nu d)$	distance from the origin.
$A = \tau/(1-c)$ (when $\tau \to 0$, $c \to 1$).	
A_2	defined in equation (66).
$B = n/r$.	
C	capacitance per unit length.
$D = nd = vT$.	
F	$F(y)$ is the probability that the coordinate of a particle relative to the origin is $\leqslant y$.
F^*	the limit of F in the continuous distribution (see equation (95)).
$G = \tau/(1-p-q)$ when there is leakage (section 10).	
I_0, I_1	Bessel functions of imaginary argument (equation (81)).
K	a constant ($= \nu/n$ in (63) and in section 6).
K_1	a constant.
L	for the meaning of L in equation (66), see equation (65). In sections 8, 9, 10, L is self-inductance per unit length.
M	$M(n,u)$ is the moment-generating function $\sum_{\nu=-n}^{n} \gamma(n,\nu)e^{i\nu u}$.
N	half the total number of particles (section 8).
R	resistance per unit length.
S	$1/S$ is the leakage resistance per unit length.
S_1, S_2, S_3, S_4, S_5	integrals defined in equations (42), (43), (44), (54), and (60).
$T = n\tau$	(from section 8 onwards, t is used in place of T.)
U_n	Lommel's function of two variables (ref. **5**, sections 16.5–16.59, pp. 537–50).
V	potential.

$W = (vt-y)/(2Av)$.

$W_1, W_2 = \frac{1}{2}\left(\frac{1}{A^{\frac{1}{2}}} \pm \frac{1}{G^{\frac{1}{2}}}\right)^2 \left(t - \frac{y}{v}\right)$.

$Y = B(1-\nu^2/n^2)^{\frac{1}{2}} = B(1-K^2)^{\frac{1}{2}} = B(1-y^2/D^2)^{\frac{1}{2}}$ ($= (v^2t^2-y^2)^{\frac{1}{2}}/(2Av)$ in the limit). In section 10, Y is the positive square root of $\frac{1}{4}\left(\frac{1}{A} - \frac{1}{G}\right)^2 \left(t^2 - \frac{y^2}{v^2}\right)$.

$\alpha(n, \nu)$, $\beta(n, \nu)$ the fraction of the number of particles at a distance νd from the origin after a time $n\tau$, and arriving from the left ($\alpha(n, \nu)$) and from the right ($\beta(n, \nu)$), respectively.

$\gamma(n, \nu)$ ($= \alpha(n, \nu)+\beta(n, \nu)$): the total fraction of the number of particles at a distance νd from the origin after a time $n\tau$.

θ for definition, see equation (55).

ϑ temperature.

κ thermometric conductivity.

ν an integer.

$\rho = 1/r = q/p$.

σ line density.

τ time interval to describe a partial path.

ϕ for definition, see equation (56).

χ $y = D \sin \chi$, $Y = B \cos \chi$ (equation (91)).

$\psi = \sin^{-1}(r \sin u)$.

3. The difference equation for the density of the distribution. The solution for the moment-generating function

Let p be the probability that a particle will continue to move in the same direction at the end of a partial path, and $q = 1-p$ the probability that the direction of its velocity will be reversed. The correlation coefficient is

$$c = p-q. \tag{3}$$

Let $\gamma(n, \nu)$ be the fraction of the number of particles at a distance νd from the origin after a time $n\tau$, $\alpha(n, \nu)$ being the fraction that, moving towards the right (in the positive direction), arrives from the left, and $\beta(n, \nu)$ the fraction that arrives from the right, moving towards the left. (n is positive; ν may be positive, negative, or zero.) Then, by elementary reasoning,

$$\gamma(n, \nu) = \alpha(n, \nu)+\beta(n, \nu), \tag{4}$$

$$\alpha(n, -\nu) = \beta(n, \nu), \qquad \gamma(n, -\nu) = \gamma(n, \nu), \tag{5}$$

$$\alpha(n, \nu) = \beta(n, \nu) = \gamma(n, \nu) = 0 \quad \text{when } n-\nu \text{ is odd, when } \nu < -n, \text{ and when } \nu > n; \tag{6}$$

$$\left.\begin{array}{c}\alpha(1,1)=\beta(1,-1)=\tfrac{1}{2}, \qquad \alpha(1,-1)=\beta(1,1)=0,\\ \gamma(1,1)=\gamma(1,-1)=\tfrac{1}{2}\end{array}\right\}, \tag{7}$$

$$\left.\begin{array}{c}\alpha(2,2)=\beta(2,-2)=\tfrac{1}{2}p, \qquad \alpha(2,0)=\beta(2,0)=\tfrac{1}{2}q,\\ \alpha(2,-2)=\beta(2,2)=0,\\ \gamma(2,2)=\gamma(2,-2)=\tfrac{1}{2}p, \qquad \gamma(2,0)=q\end{array}\right\}; \tag{8}$$

and the following recurrence relations hold:

$$\alpha(n+1,\nu)=p\alpha(n,\nu-1)+q\beta(n,\nu-1)=p\gamma(n,\nu-1)-c\beta(n,\nu-1), \tag{9}$$

$$\beta(n+1,\nu)=p\beta(n,\nu+1)+q\alpha(n,\nu+1)=p\gamma(n,\nu+1)-c\alpha(n,\nu+1). \tag{10}$$

From (9) and (10),

$$\begin{aligned}\alpha(n,\nu+1)+\beta(n,\nu-1)&=p\alpha(n-1,\nu)+q\beta(n-1,\nu)+p\beta(n-1,\nu)+q\alpha(n-1,\nu)\\ &=\gamma(n-1,\nu).\end{aligned} \tag{11}$$

Hence, by addition of (9) and (10),

$$\gamma(n+1,\nu)=p[\gamma(n,\nu-1)+\gamma(n,\nu+1)]-c\gamma(n-1,\nu). \tag{12}$$

This is the difference equation for $\gamma(n,\nu)$. Now define

$$M(n,u)=\sum_{\nu=-n}^{n}\gamma(n,\nu)e^{i\nu u}=\gamma(n,0)+2\sum_{\nu=1}^{n}\gamma(n,\nu)\cos\nu u, \tag{13}$$

where u is a real variable. From (7) and (8),

$$M(1,u)=\cos u, \qquad M(2,u)=q+p\cos 2u=1-2p\sin^2 u. \tag{14}$$

Multiply (12) by $e^{i\nu u}$ and sum for all values of ν from $-(n+1)$ to $n+1$. Since $\gamma(n,\nu)=0$ for $\nu<-n$ and for $\nu>n$, the result is

$$\begin{aligned}M(n+1,u)&=pe^{iu}\sum_{\nu-1=-n}^{\nu-1=n}\gamma(n,\nu-1)e^{i(\nu-1)u}+\\ &\quad+pe^{-iu}\sum_{\nu+1=-n}^{\nu+1=n}\gamma(n,\nu+1)e^{i(\nu+1)u}-c\sum_{\nu=-(n-1)}^{\nu=n-1}\gamma(n-1,\nu)e^{i\nu u}\\ &=p(e^{iu}+e^{-iu})M(n,u)-cM(n-1,u)\\ &=2p\cos uM(n,u)-cM(n-1,u).\end{aligned} \tag{15}$$

This is the difference equation for $M(n,u)$. Its solution is

$$M(n,u)=k_1a_1^n+k_2a_2^n, \tag{16}$$

where a_1 and a_2 are the roots of the quadratic equation

$$x^2-2px\cos u+c=0,$$

i.e. with $\quad r=\dfrac{p}{q}, \qquad p=\dfrac{r}{r+1}, \qquad q=\dfrac{1}{r+1}, \qquad c=\dfrac{r-1}{r+1}, \qquad (17)$

$$a_1=p\cos u+q\{1-r^2\sin^2u\}^{\frac{1}{2}}, \qquad a_2=p\cos u-q\{1-r^2\sin^2u\}^{\frac{1}{2}}. \tag{18}$$

(This result may also be obtained by manipulation of the transition matrix of the Markoff chain: cf. ref. **3**.)

The values of k_1 and k_2 are found from those of $M(1,u)$ and $M(2,u)$ in (14), and are

$$k_1 = \frac{1}{2}\left[1+\frac{\cos u}{(1-r^2\sin^2 u)^{\frac{1}{2}}}\right], \qquad k_2 = \frac{1}{2}\left[1-\frac{\cos u}{(1-r^2\sin^2 u)^{\frac{1}{2}}}\right]. \tag{19}$$

The moments of the distribution about the origin are found from the expansion of $M(n,u)$ in ascending powers of u; the odd moments are zero, and $\sum\limits_{\nu=-n}^{n} \nu^{2m}\gamma(n,\nu)$ is $(-1)^m$ times the coefficient of $u^{2m}/(2m)!$ in the expansion of $M(n,u)$. For example,

$$\sum \nu^2\gamma(n,\nu) = nr-\tfrac{1}{2}(r^2-1)(1-c^n) \tag{20}$$

(in agreement with (1)) and

$$\sum \nu^4\gamma(n,\nu) = 3n^2r^2-6nr^3+4nr+\tfrac{1}{2}(r^2-1)(9r^2-1)(1-c^n)-3nr(r^2-1)c^n. \tag{21}$$

If we carry out the limiting operation $\tau \to 0$ with $n\tau = T$, $\tau/(1-c) = A$, as before, then $r\tau \to 2A$, and we have, as before, for the mean square distance from the origin,

$$d^2 \sum \nu^2\gamma(n,\nu) \to v^2[2AT-2A^2(1-e^{-T/A})]; \tag{22}$$

also

$$d^4 \sum \nu^4\gamma(n,\nu) \to 12v^4[A^2T^2-2A^3T(2+e^{-T/A})+6A^4(1-e^{-T/A})], \tag{23}$$

and we verify that this becomes v^4T^4 for small T, as it should.

It follows from (13) that, in terms of $M(n)$,

$$\gamma(n,\nu) = \frac{1}{\pi}\int\limits_0^{\pi} M(n,u)\cos\nu u\,du. \tag{24}$$

Since

$$\left.\begin{aligned} a_1(\pi-u) = -a_2(u), \qquad k_1(\pi-u) = k_2(u), \\ M(n,\pi-u) = (-1)^n M(n,u) \end{aligned}\right\}, \tag{25}$$

$\gamma(n,\nu) = 0$ if $n-\nu$ is odd, as it should, and if $n-\nu$ is even

$$\gamma(n,\nu) = \frac{2}{\pi}\int\limits_0^{\frac{1}{2}\pi} M(n,u)\cos\nu u\,du = \frac{2}{\pi}\int\limits_0^{\pi} k_1 a_1^n \cos\nu u\,du. \tag{26}$$

4. The computation of $\gamma(n,\nu)$. Asymptotic expansions for n large, r fixed

The values of γ for $n = 1$ and 2 are given in (7) and (8), and the recurrence relation in (12); for small values of n it is easy to write down the values of γ explicitly. For any value of n,

$$\gamma(n,n) = \tfrac{1}{2}p^{n-1}, \tag{27}$$

and for an integral $s > 0$ and $\leqslant \frac{1}{2}n$,

$$\gamma(n, n-2s) = \frac{r^{n-3}(r-1)}{(r+1)^{n-1}}\left[1+(s-1)(n-s-1)\frac{1}{r^2}+\frac{(s-1)_2}{2!}\frac{(n-s-1)_2}{2!}\frac{1}{r^4}+\right.$$
$$\left.+\frac{(s-1)_3}{3!}\frac{(n-s-1)_3}{3!}\frac{1}{r^6}+\dots\right]+$$
$$+\frac{nr^{n-3}}{2(r+1)^{n-1}}\left[1+\tfrac{1}{2}(s-1)(n-s-1)\frac{1}{r^2}+\frac{1}{3}\frac{(s-1)_2}{2!}\frac{(n-s-1)_2}{2!}\frac{1}{r^4}+\right.$$
$$\left.+\frac{1}{4}\frac{(s-1)_3}{3!}\frac{(n-s-1)_3}{3!}\frac{1}{r^6}+\dots\right], \tag{28}$$

where
$$(x)_m = x(x-1)\dots(x-m+1). \tag{29}$$

If s is not large, the values of γ are easily found from this formula for any n.

When $r = 1$ the problem is the classical one of random flights in one dimension; the solution is

$$\gamma(n,\nu) = \frac{1}{2^n}\frac{n!}{\{\frac{1}{2}(n-\nu)\}!\,\{\frac{1}{2}(n+\nu)\}!}, \tag{30}$$

with the asymptotic expansion (**4**) for large n and $\nu^2 = O(n)$ or $o(n)$

$$\gamma(n,\nu) \sim \left(\frac{2}{\pi n}\right)^{\frac{1}{2}} e^{-\nu^2/2n} \times$$
$$\times\left\{1-\frac{1}{4n}\left(1-2\frac{\nu^2}{n}+\frac{1}{3}\frac{\nu^4}{n^2}\right)+\frac{1}{32n^2}\left(1-\frac{44}{3}\frac{\nu^2}{n}+\frac{38}{3}\frac{\nu^4}{n^2}-\frac{12}{5}\frac{\nu^6}{n^3}+\frac{1}{9}\frac{\nu^8}{n^4}\right)+\dots\right\}. \tag{31}$$

When $r \neq 1$ an asymptotic expansion similar to (31) may be found from the integral (26). ((28) may be expressed in terms of hypergeometric functions, but it seems simpler to use (26).) The result is

$$\gamma(n,\nu) \sim \left(\frac{2}{\pi rn}\right)^{\frac{1}{2}} e^{-\nu^2/2rn}\left\{1-\frac{1}{4rn}\left(\frac{r^2+1}{2}-2r\frac{\nu^2}{n}+\frac{3r^2-1}{6r^2}\frac{\nu^4}{n^2}\right)-\right.$$
$$-\frac{1}{32r^2n^2}\left(\frac{3r^4-10r^2+3}{4}+\frac{18r^4+30r^2-4}{3r}\frac{\nu^2}{n}-\frac{81r^4-10r^2+5}{6r^2}\frac{\nu^4}{n^2}+\right.$$
$$\left.\left.+\frac{20r^4-10r^2+2}{5r^3}\frac{\nu^6}{n^3}-\frac{(3r^2-1)^2}{36r^4}\frac{\nu^8}{n^4}\right)+\dots\right\}. \tag{32}$$

A comparison of numerical values will be set out before a brief description of the derivation of the formula.

For numerical comparison, n is taken as 15, and r as $\frac{1}{2}$, 1, 2, and 5. In Table 1, each column headed I contains the correct values of $\gamma(15,\nu)$; the column headed II contains 10^7 times the error in the asymptotic expansion (32) taken as far as the terms shown there explicitly—i.e. 10^7 times

(column 1 minus values calculated from the terms shown explicitly on the right-hand side of (32)).

TABLE 1

	$r=\frac{1}{2}, c=-\frac{1}{3}$		$r=1, c=0$		$r=2, c=\frac{1}{3}$		$r=5, c=\frac{2}{3}$	
ν	I	II	I	II	I	II	I	II
1	0·2674364	−5	0·1963806	+25	0·1405597	−20	$0{\cdot}0874406_5$	−825
3	0·1600470	+77	0·1527405	−17	0·1249851	−80	0·0842032	−1161
5	0·0578812	−94	0·0916443	−72	0·0983322	−170	0·0779941	−2026
7	0·0127762	−9	0·0416565	+53	0·0676935	+159	0·0693163	−2974
9	0·0017169	+41	0·0138855	+69	0·0399551	+788	0·0588573	−2482
11	0·0001363	−2	0·0032043	−47	0·0194825	+613	0·0474173	+1091
13	$0{\cdot}0000058_5$	−4	0·0004578	−35	0·0072792	−554	0·0358278	+7695
15	0·0000001	−2	0·0000305	+14	0·0017127	−1311	0·0389433	+155391

For the values shown in the table the accuracy of (32) decreases, for a fixed n, as r increases from 1. The formula is not satisfactory, for any n, if r is too small or too large. The case of small r is not further considered here. For large r and large n the present method would not normally be used (see section 6 and Table 2).

The derivation of (32) from (26) is fairly straightforward, but a satisfactory estimation of the error terms is rather tedious, since the methods to be used and, to some extent, the results, depend on the value of r. Careful estimations of the error terms appear to be of no great interest, and are, for the most part, omitted; only a brief sketch of the proof of (32) will be given.

Let us begin with the case $r < 1$, and use the second expression in (26). For $r < 1$, a_1 is real and positive, and decreases from 1 to $-c$ as u increases from 0 to π, where $-c = (1-r)/(1+r)$; k_1 is positive and decreases from 1 to 0, so $k_1 a_1^n$ also is positive and decreases from 1 to 0.

We may, for a certain range of values of u, expand $\log a_1 + \frac{1}{2}ru^2$ in powers of u:

$$\log a_1 + \tfrac{1}{2}ru^2 = -\tfrac{1}{24}r(3r^2-1)u^4 - \tfrac{1}{720}r(45r^4-30r^2+1)u^6 + O(u^8).$$

We may find an expression for $a_1^n e^{\frac{1}{2}nru^2}$ by expanding $\exp\{n(\log a_1 + \frac{1}{2}ru^2)\}$ in powers of n; if error terms are included, we may then substitute for $\log a_1 + \frac{1}{2}ru^2$. To include satisfactory error terms we require that

$$\exp\{n(\log a_1 + \tfrac{1}{2}ru^2)\}$$

should be bounded, so we restrict u to the range $0 \leqslant u \leqslant K_1 n^{-\frac{1}{4}}$, where K_1 is a constant; that $\exp\{n(\log a_1 + \frac{1}{2}ru^2)\}$ is then bounded follows from

$$\frac{d}{du}(\log a_1) = -\frac{r\sin u}{(1-r^2\sin^2 u)^{\frac{1}{2}}} < -r\sin u < -ru + \frac{ru^3}{6}, \tag{33}$$

so
$$\log a_1+\tfrac{1}{2}ru^2 < \frac{ru^4}{24} \leqslant \frac{rK_1^4}{24n}. \tag{34}$$
We may now use finite expansions with error terms to show that, if
$$w = (nr)^{\frac{1}{2}}u, \tag{35}$$
then
$$k_1 a_1^n e^{\frac{1}{2}w^2} = 1+\frac{w^2}{4nr}(r^2-1)-\frac{w^4}{24nr}(3r^2-1)+\frac{w^4}{48n^2r^2}(r^2-1)(9r^2-1)-$$
$$-\frac{w^6}{1440n^2r^2}(135r^4-120r^2+1)+\frac{w^8}{1152n^2r^2}(3r^2-1)^2+..., \tag{36}$$
with errors of orders w^6/n^3, w^8/n^3, w^{10}/n^3, w^{12}/n^3, and with w restricted to the range $0 \leqslant w \leqslant K_1 r^{\frac{1}{2}} n^{\frac{1}{4}}$.

The whole of the asymptotic expansion comes from
$$\frac{2}{\pi}\int\limits_0^{K_1 n^{-\frac{1}{4}}} k_1 a_1^n \cos \nu u\, du; \tag{37}$$
$k_1 < 1$ and a_1 is a decreasing function of u, so in the rest of the range of integration the integrand has a modulus less than the value of a_1^n at $u = K_1 n^{-\frac{1}{4}}$, which in turn is less than $\exp(rK_1^4/24)\exp(-\frac{1}{2}rK_1^2 n^{\frac{1}{2}})$; and the error due to neglecting the rest of the integral is less than twice this.

In (37) we now substitute from (36), and change the variable of integration to w. The range of integration is then from 0 to $K_1 r^{\frac{1}{2}} n^{\frac{1}{4}}$. The range is extended to be from 0 to ∞, the error so introduced being of order $n^{-\frac{1}{4}}\exp(-\frac{1}{2}rK_1^2 n^{\frac{1}{2}})$ at most. The value of $k_1 a_1^n$ from (36), multiplied by $\cos[\nu w/(nr)^{\frac{1}{2}}]$, is integrated term by term, the equality
$$\int\limits_0^\infty e^{-\frac{1}{2}w^2} w^{2m} \cos\frac{\nu w}{(nr)^{\frac{1}{2}}}\, dw = e^{-\frac{1}{2}\nu^2/nr}\,\mathrm{re}\int\limits_0^\infty e^{-\frac{1}{2}x^2}\left[x+\frac{i\nu}{(nr)^{\frac{1}{2}}}\right]^{2m} dx \tag{38}$$
being used. The error in stopping after a finite number of terms is of the order of the first neglected term. Thus (32) is obtained for $r < 1$.

For $r = 1$, k_1 is 1 in $(0, \frac{1}{2}\pi)$ and 0 in $(\frac{1}{2}\pi, \pi)$; a_1 is $\cos u$ in $(0, \frac{1}{2}\pi)$ and 0 in $(\frac{1}{2}\pi, \pi)$. Equation (30) and Stirling's formula lead at once to the result (31), but the calculations from the integral could proceed as before; however, in the expansion of $\exp\{n(\log a_1+\frac{1}{2}u^2)\}$, u need not be restricted to an interval $(0, K_1 n^{-\frac{1}{4}})$, but may be allowed to have any value in an interval $(0, \frac{1}{2}\pi-\delta)$ for any positive δ, since $\log a_1+\frac{1}{2}u^2$ is negative, and the exponential therefore bounded, in the whole of that range. The error terms are correspondingly altered. [A somewhat similar remark applies to a certain range of values of $r < 1$; if r is not too small, there is an interval of u, independent of n, in which $\log a_1+\frac{1}{2}ru^2$ is negative, which may replace the interval $(0, K_1 n^{-\frac{1}{4}})$.]

Consider now the case $r > 1$, which is the case of greatest interest. Let

$$u_1 = \sin^{-1}\frac{1}{r}. \tag{39}$$

Then k_1 and k_2 are infinite at $u = u_1$; k_1 increases from 1, and k_2 decreases from 0, at $u = 0$; a_1 and a_2 are finite, and each is equal to $c^{\frac{1}{2}}$, where

$$c = (r-1)/(r+1),$$

at $u = u_1$; a_1 decreases from 1, and a_2 increases from c, at $u = 0$, to this value. $M(n, u)$ is finite and equal to $c^{\frac{1}{2}n}(1+n/r)$ at $u = u_1$; it decreases to this value from 1 at $u = 0$.

Between u_1 and $\frac{1}{2}\pi$, a_1 and a_2 are conjugate complex, and so are k_1 and k_2, with

$$a_1 = \frac{1}{r+1}[r\cos u + i(r^2\sin^2 u - 1)^{\frac{1}{2}}], \qquad k_1 = \frac{1}{2}\left[1 - i\,\frac{\cos u}{(r^2\sin^2 u - 1)^{\frac{1}{2}}}\right]; \tag{40}$$

$M(n, u)$ is oscillatory, and it may be proved by induction that $M(2n, u)$, $M(2n+1, u)/(\cos u)$, are both polynomials of degree n in $\cos^2 u$, each with n zeros between $u = u_1$ and $u = \frac{1}{2}\pi$.

From (26),

$$\tfrac{1}{2}\pi\gamma(n, \nu) = S_1 + S_2 + S_3, \tag{41}$$

where

$$S_1 = \int_0^{u_1} k_1 a_1^n \cos \nu u\, du, \tag{42}$$

$$S_2 = \int_0^{u_1} k_2 a_2^n \cos \nu u\, du, \tag{43}$$

and

$$S_3 = \int_{u_1}^{\frac{1}{2}\pi} (k_1 a_1^n + k_2 a_2^n)\cos \nu u\, du. \tag{44}$$

Write

$$\rho = \frac{1}{r}, \tag{45}$$

$$\psi = \sin^{-1}(r \sin u), \tag{46}$$

$$l_1 = \frac{1}{2}\left[1 + \frac{\cos\psi}{(1-\rho^2\sin^2\psi)^{\frac{1}{2}}}\right], \qquad l_2 = \frac{1}{2}\left[1 - \frac{\cos\psi}{(1-\rho^2\sin^2\psi)^{\frac{1}{2}}}\right], \tag{47}$$

$$b_1 = \frac{1}{1+\rho}[\rho\cos\psi + (1-\rho^2\sin^2\psi)^{\frac{1}{2}}], \qquad b_2 = \frac{1}{1+\rho}[\rho\cos\psi - (1-\rho^2\sin^2\psi)^{\frac{1}{2}}] \tag{48}$$

(so that l_1, l_2, b_1, b_2, are the same functions of ρ and ψ as k_1, k_2, a_1, a_2 are of r and u); then

$$k_1\, du = \rho l_1\, d\psi, \qquad k_2\, du = -\rho l_2\, d\psi, \qquad a_1 = b_1, \qquad a_2 = -b_2, \tag{49}$$

and

$$S_1 = \rho \int\limits_0^{\frac{1}{2}\pi} l_1 b_1^n \cos\{\nu \sin^{-1}(\rho \sin\psi)\}\, d\psi, \tag{50}$$

$$S_2 = (-1)^{n+1}\rho \int\limits_0^{\frac{1}{2}\pi} l_2 b_2^n \cos\{\nu \sin^{-1}(\rho \sin\psi)\}\, d\psi$$

$$= -\rho \int\limits_{\frac{1}{2}\pi}^{\pi} l_1 b_1^n \cos\{\nu \sin^{-1}(\rho \sin\psi)\}\, d\psi. \tag{51}$$

Since as ψ increases from $\frac{1}{2}\pi$ to π, $l_1 b_1^n$ decreases from its value

$$\tfrac{1}{2}[(1-\rho)/(1+\rho)]^{\frac{1}{2}n}$$

at $\frac{1}{2}\pi$ to zero at π, $|S_2| < \frac{1}{2}\pi\rho[(1-\rho)/(1+\rho)]^{\frac{1}{2}n}$. We may now write a transformation of the integral, which we shall use later. If

$$l_3 = 1 - \frac{\sin\psi}{(1-\rho^2\cos^2\psi)^{\frac{1}{2}}}, \qquad b_3 = \frac{(1-\rho^2\cos^2\psi)^{\frac{1}{2}} - \rho\sin\psi}{(1-\rho^2)^{\frac{1}{2}}}, \tag{52}$$

then

$$S_2 = -\tfrac{1}{2}\rho\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}n} S_4, \tag{53}$$

where

$$S_4 = \int\limits_0^{\frac{1}{2}\pi} l_3 b_3^n \cos\{\nu \sin^{-1}(\rho\cos\psi)\}\, d\psi. \tag{54}$$

S_2 may be omitted in establishing the asymptotic expansion (32). To consider S_3, commence by putting

$$\theta = \tan^{-1}\frac{(r^2\sin^2 u - 1)^{\frac{1}{2}}}{r\cos u} \qquad (= \arg a_1), \tag{55}$$

$$\phi = \tan^{-1}\frac{\cos u}{(r^2\sin^2 u - 1)^{\frac{1}{2}}} = \tan^{-1}\left(\frac{1}{r}\cot\theta\right) \qquad (= -\arg k_1). \tag{56}$$

Then, from (40),

$$S_3 = \left(\frac{r-1}{r+1}\right)^{\frac{1}{2}n} (r^2-1)^{\frac{1}{2}} \int\limits_{u_1}^{\frac{1}{2}\pi} \frac{\sin u}{(r^2\sin^2 u - 1)^{\frac{1}{2}}} \cos(n\theta - \phi)\cos\nu u\, du. \tag{57}$$

Now $\qquad (r^2-1)\cos^2\theta = r^2\cos^2 u, \qquad (r^2-1)\sin^2\theta = r^2\sin^2 u - 1, \qquad (58)$

so if the variable of integration is changed to θ,

$$S_3 = \left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}n} (1-\rho^2)^{\frac{1}{2}} S_5, \tag{59}$$

where

$$S_5 = \int\limits_0^{\frac{1}{2}\pi} \cos\{n\theta - \tan^{-1}(\rho\cot\theta)\}\cos\{\nu\cos^{-1}[(1-\rho^2)^{\frac{1}{2}}\cos\theta]\}\, d\theta. \tag{60}$$

$|S_5| < \frac{1}{2}\pi$, so S_3 may be omitted. The asymptotic expansion (32) therefore comes entirely from S_1. More exactly, we shall prove later that

$$\left.\begin{aligned} 2S_2 + S_3 &= 0 \quad \text{for } \nu < n \\ &= \tfrac{1}{4}\pi(1-\rho)^{n+1} \quad \text{for } \nu = n > 0 \end{aligned}\right\}, \tag{61}$$

so

$$\left.\begin{aligned}\tfrac{1}{2}\pi\gamma(n,\nu) = S_1 - S_2 &= \rho\int\limits_0^\pi l_1 b_1^n \cos\{\nu\sin^{-1}(\rho\sin\psi)\}\,d\psi \quad \text{for } \nu < n \\ &= S_1 - S_2 + \tfrac{1}{4}\pi(1-\rho)^{n+1} \quad \text{for } \nu = n > 0.\end{aligned}\right\} \tag{62}$$

The asymptotic expansion (32) may be established from (62) for $r > 1$ in the same way as it was established from (26) for $r \leqslant 1$, the only new calculation required being that $\cos^{-1}\{\nu\sin^{-1}(\rho\sin\psi)\}$, after the substitution $w = (nr)^{\frac{1}{2}}\psi$ for the variable of integration (corresponding to the substitution (35)), must be expanded in powers of n^{-1}. Alternatively, and rather more easily, the argument may start from the original integral (24), and proceed exactly as before; u may be taken over the interval $(0, u_1-\delta)$, for any positive δ, since $\log a_1 + \frac{1}{2}ru^2$ is negative; and the transformations shown above may be used to find an upper bound to the error in neglecting all but the integral, from 0 to $u_1-\delta$, of the integrand in (42).

The very large jump in the error in Table 1 for $r = 5$, between $\nu = 13$ and $\nu = 15$, is largely accounted for by the term $\frac{1}{4}\pi(1-\rho)^{n+1}$ in (62).

The asymptotic expansion (32) is valid for $\nu^2 = O(n)$ or $o(n)$; when ν is nearly equal to n, direct computation from (28) is not difficult; when ν/n is equal to a constant K, not nearly equal to 1, then from (30), for $r = 1$,

$$\gamma(n,\nu) \sim \left(\frac{2}{n\pi}\right)^{\frac{1}{2}}(1-K^2)^{-\frac{1}{2}(n+1)}\left(\frac{1-K}{1+K}\right)^{\frac{1}{2}Kn}\left\{1 - \frac{3+K^2}{12(1-K^2)}\frac{1}{n} + \dots\right\}, \tag{63}$$

and for $r \neq 1$ this formula may be generalized, by use of the method of steepest descents on the integrals in (26) and (62), to

$$\begin{aligned}\gamma(n,\nu) \sim \left(\frac{2}{n\pi}\right)^{\frac{1}{2}} L^{-\frac{1}{2}} r^{Kn}\left(\frac{L+r}{2r}\right)\left(\frac{1+L}{1+r}\right)^n \times \\ \times (1-K^2)^{-\frac{1}{2}(n+1)}\left[\frac{(1-K^2)^{\frac{1}{2}}}{L+K}\right]^{Kn}\left\{1 - \frac{A_2}{n} + \dots\right\},\end{aligned} \tag{64}$$

where

$$L = \{K^2 + r^2(1-K^2)\}^{\frac{1}{2}} \tag{65}$$

and

$$\begin{aligned}A_2 = \frac{3r^2(3r^2-1) - K^2(18r^4-18r^2-2) + K^4(9r^4-15r^2+6)}{24(1-K^2)L^3} - \\ - \frac{r^2(r^2-1)(1-K^2)}{2L^2(L+r)}.\end{aligned} \tag{66}$$

5. Proof of the relation between the integrals S_2 and S_3

To prove (61), with $\rho = 1/r < 1$, it follows from (53) and (59) that it is necessary and sufficient to prove that

$$\left.\begin{aligned}S_5 - \frac{\rho}{(1-\rho^2)^{\frac{1}{2}}}S_4 &= 0 \quad \text{for } \nu < n \\ &= \tfrac{1}{4}\pi(1-\rho)(1-\rho^2)^{\frac{1}{2}(n-1)} \quad \text{for } \nu = n > 0\end{aligned}\right\}. \tag{67}$$

S_5 is given by (60), and it is easily seen that

$$S_5 = \tfrac{1}{2}\,\mathrm{re}\int\limits_0^{\frac{1}{2}\pi} e^{in\theta}\frac{\sin\theta - i\rho\cos\theta}{(\rho^2\cos^2\theta+\sin^2\theta)^{\frac{1}{2}}}\{[(1-\rho^2)^{\frac{1}{2}}\cos\theta + i(\rho^2\cos^2\theta+\sin^2\theta)^{\frac{1}{2}}]^\nu + \\ +[(1-\rho^2)^{\frac{1}{2}}\cos\theta - i(\rho^2\cos^2\theta+\sin^2\theta)^{\frac{1}{2}}]^\nu\}\,d\theta. \quad (68)$$

S_4 is given by (54). In the integral, substitute

$$(1-\rho^2)^{\frac{1}{2}}\sinh x_1 = \rho\sin\psi, \qquad (1-\rho^2)^{\frac{1}{2}}\cosh x_1 = (1-\rho^2\cos^2\psi)^{\frac{1}{2}}. \quad (69)$$

Then

$$\left.\begin{aligned} b_3 = e^{-x_1}, \qquad \frac{\rho}{(1-\rho^2)^{\frac{1}{2}}}l_3\,d\psi = \left(\frac{\rho\cosh x_1-\sinh x_1}{\rho\cosh x_1+\sinh x_1}\right)^{\frac{1}{2}} \\ \exp[i\sin^{-1}(\rho\cos\psi)] = (1-\rho^2)^{\frac{1}{2}}\cosh x_1 + i(\rho^2\cosh^2 x_1-\sinh^2 x_1)^{\frac{1}{2}} \end{aligned}\right\}, \quad (70)$$

and as ψ increases from 0 to $\frac{1}{2}\pi$, x_1 increases from 0 to $\tanh^{-1}\rho$, so

$$\frac{\rho}{(1-\rho^2)^{\frac{1}{2}}}S_4 = \frac{1}{2}\int\limits_0^{\tanh^{-1}\rho} e^{-nx_1}\left(\frac{\rho\cosh x_1-\sinh x_1}{\rho\cosh x_1+\sinh x_1}\right)^{\frac{1}{2}}\{[(1-\rho^2)^{\frac{1}{2}}\cosh x_1 + \\ + i(\rho^2\cosh^2 x_1-\sinh^2 x)^{\frac{1}{2}}]^\nu + [(1-\rho^2)^{\frac{1}{2}}\cosh x_1 - i(\rho^2\cosh^2 x_1-\sinh^2 x_1)^{\frac{1}{2}}]^\nu\}\,dx_1. \quad (71)$$

To prove (67) consider

$$-\frac{1}{2}\int e^{nz}\frac{\rho\cosh z+\sinh z}{(\rho^2\cosh^2 z-\sinh^2 z)^{\frac{1}{2}}}\{[(1-\rho^2)^{\frac{1}{2}}\cosh z + i(\rho^2\cosh^2 z-\sinh^2 z)^{\frac{1}{2}}]^\nu + \\ +[(1-\rho^2)^{\frac{1}{2}}\cosh z - i(\rho^2\cosh^2 z-\sinh^2 z)^{\frac{1}{2}}]^\nu\}\,dz, \quad (72)$$

where $z = x+iy$, round the contour shown in Fig. 1. AB is $x = 0$, $0 \leqslant y \leqslant \frac{1}{2}\pi$; BC is $y = \frac{1}{2}\pi$, $0 \geqslant x \geqslant -X$, where $X \to \infty$; CD is at $x = -X$; AED is along the negative real axis, and is indented at E, by a small semicircle whose radius $\to 0$, where E is at $x = -\tanh^{-1}\rho$. Then

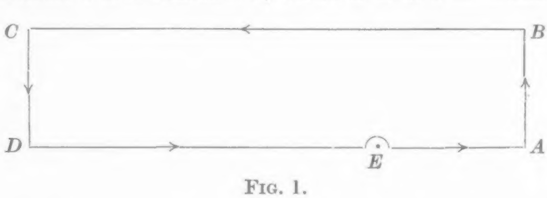

FIG. 1.

the real part of the integral along AB is S_5; when, as here, $\frac{1}{2}(n-\nu)$ is an integer, the integral along BC is purely imaginary; the integral along $CD \to 0$ for $\nu < n$ and $\to -\frac{1}{4}\pi(1-\rho)(1-\rho^2)^{\frac{1}{2}(n-1)}$ for $\nu = n$; the integral along DE is purely imaginary; the integral round the small semicircle at $E \to 0$; and the integral along EA is $-\rho S_4/(1-\rho^2)^{\frac{1}{2}}$, according to (71). Hence (67), and (61), are proved.

6. The asymptotic expression for large n, n/r fixed

Consider next the values of $\gamma(n,\nu)$ for large values of n and r, when r/n is constant, and equal to $1/B$, say. Thus as $n\to\infty$, $r\to\infty$, $c\to 1$. Also let ν/n be constant, and equal to K; then K is the ratio of the distance $y=\nu d$ from the origin to the greatest distance $D=vT$, where v is the particle velocity and T the total time of motion, equal to $n\tau$ or nd/v. Note also that to accord with Taylor's notation, in which $\tau/(1-c)=A$, we must have

$$B=\frac{T}{A(1+c)}. \tag{73}$$

As $n\to\infty$, we may suppose that $\tau\to 0$ and $d\to 0$ in such a way that $T=n\tau$ and $D=nd$ remain finite. Then in the limit the mean square distance of the particles from the origin is given by

$$\frac{\overline{y^2}}{v^2T^2}=\frac{1}{B}-\frac{1}{2B^2}(1-e^{-2B}). \tag{74}$$

(Cf. (2) and (22).) $\gamma(n,\nu)$ is given by (62), $\rho=1/r=B/n$, and as $n\to\infty$

$$\left.\begin{aligned} l_1\to\tfrac{1}{2}(1+\cos\psi),\qquad & b_1^n\to\exp(-B+B\cos\psi),\\ \cos\{\nu\sin^{-1}(\rho\sin\psi)\}&\to\cos(KB\sin\psi)\end{aligned}\right\}. \tag{75}$$

The integrand tends uniformly to its limit and

$$nS_1\to\tfrac{1}{2}Be^{-B}\int\limits_0^{\frac{1}{2}\pi}(1+\cos\psi)\exp(B\cos\psi)\cos(KB\sin\psi)\,d\psi. \tag{76}$$

Similarly,

$$nS_2\to\tfrac{1}{2}Be^{-B}\int\limits_0^{\frac{1}{2}\pi}(1-\cos\psi)\exp(-B\cos\psi)\cos(KB\sin\psi)\,d\psi. \tag{77}$$

Hence, for $\nu<n$,

$$\tfrac{1}{2}\pi n\gamma(n,\nu)\to Be^{-B}\int\limits_0^{\frac{1}{2}\pi}\{\cosh(B\cos\psi)+\cos\psi\sinh(B\cos\psi)\}\cos(KB\sin\psi)\,d\psi. \tag{78}$$

Since†

$$\frac{2}{\pi}\int\limits_0^{\frac{1}{2}\pi}\cos(KB\sin\psi)\cosh(B\cos\psi)\,d\psi=I_0(Y), \tag{79}$$

$$\frac{2}{\pi}\int\limits_0^{\frac{1}{2}\pi}\cos(KB\sin\psi)\sinh(B\cos\psi)\cos\psi\,d\psi=BI_1(Y)/Y, \tag{80}$$

where I_0, I_1 are the 'Bessel functions of imaginary argument',

$$I_0(z)=\sum_{m=0}^{\infty}\frac{(\frac{1}{2}z)^{2m}}{(m!)^2},\qquad I_1(z)=\sum_{m=0}^{\infty}\frac{(\frac{1}{2}z)^{2m+1}}{m!\,(m+1)!}, \tag{81}$$

† These are special cases of Sonine's second finite integral. See ref. **5**, p. 376, section 12.13.

and also
$$Y = B(1-K^2)^{\frac{1}{2}} = B(1-y^2/D^2)^{\frac{1}{2}}, \tag{82}$$
where
$$y = \nu d, \qquad D = nd = vT, \tag{83}$$
it follows that, for $\nu < n$,
$$n\gamma(n,\nu) \to Be^{-B}[I_0(Y)+BI_1(Y)/Y]. \tag{84}$$
For $\nu = n$, since
$$(1-\rho)^{n+1} = e^{-B}-\frac{Be^{-B}}{n}(1+\tfrac{1}{2}B)+O\left(\frac{1}{n^2}\right), \tag{85}$$
$$n[\gamma(n,n)-\tfrac{1}{2}e^{-B}]$$
$$\to Be^{-B}[I_0(Y)+BI_1(Y)/Y]_{Y=0}-\tfrac{1}{2}Be^{-B}(1+\tfrac{1}{2}B) = \tfrac{1}{2}Be^{-B}(1+\tfrac{1}{2}B). \tag{86}$$
The results for $\gamma(n,n)$, $\gamma(n,n-2)$, for example, are easily checked by direct calculation.

The results in (84) and (86), written in the form
$$\left.\begin{aligned}\gamma(n,\nu) &\sim Be^{-B}[I_0(Y)+BI_1(Y)/Y]/n && \text{for } \nu < n\\ &\sim \tfrac{1}{2}e^{-B}+\tfrac{1}{2}Be^{-B}[1+\tfrac{1}{2}B]/n && \text{for } \nu = n\end{aligned}\right\}, \tag{87}$$
where
$$Y = B(1-\nu^2/n^2)^{\frac{1}{2}}, \tag{88}$$
may be regarded as approximate computation formulae. For example, for a case considered previously, $n = 15$, $r = 5$, $B = 3$, the following table shows 10^6 times the error in using (87)—i.e. 10^6 times (the correct values of $\gamma(n,\nu)$ minus the values calculated from the terms shown in (87)). A comparison of the values of the errors in Table 1 and 2 shows that for this case (87), up to and including terms of order $1/n$, is more accurate than (32) up to and including terms of order $1/n^2$ inside the curly brackets.

TABLE 2

ν	1	3	5	7	9	11	13	15
10^6(error)	−114	−116	−118	−117	−60	−93	−64	+1603

7. Passage to a continuous distribution. Connexion with the limit of the characteristic function

Still pursuing a mathematical, and not a physical, investigation, we may now consider the limit of the discontinuous distribution as a continuous one, except at $y = \pm D$, where the particles heap up, with a finite fraction, $\frac{1}{2}e^{-B}$, of the total number at each of these points. Since, in the discontinuous distribution, the particles are at nd, $(n-2)d$, $(n-4)d$,..., we must, in the continuous distribution, take $2d$ as the differential, dy, of the distance y from the origin; in (84) or (87) we therefore replace $1/n$ by $dy/(2D)$, and the proportion of the total number of particles between y and $y+dy$ from the origin after a time T is
$$\frac{B}{2D}e^{-B}\left[I_0(Y)+B\frac{I_1(Y)}{Y}\right]dy. \tag{89}$$

Including the fraction $\frac{1}{2}e^{-B}$ at each end, $\frac{1}{2}$ of the particles must be on each side of the origin, so we are led to the formula

$$\frac{B}{2D}e^{-B}\int_{0}^{D}\{I_0(Y)+BI_1(Y)/Y\}\,dy = \tfrac{1}{2}(1-e^{-B}), \tag{90}$$

i.e. with

$$y = D\sin\chi, \qquad Y = B\cos\chi, \tag{91}$$

$$\int_{0}^{\frac{1}{2}\pi}[I_0(B\cos\chi)\cos\chi+I_1(B\cos\chi)]\,d\chi = (e^B-1)/B. \tag{92}$$

Similarly, from the formula for $\overline{y^2}$ in (74) we obtain

$$\int_{0}^{\frac{1}{2}\pi}[I_0(B\cos\chi)\cos\chi+I_1(B\cos\chi)]\sin^2\chi\,d\chi = \frac{e^B}{B^2}-\frac{1}{B}-\frac{\sinh B}{B^3}. \tag{93}$$

Higher moments may be considered in the same way, but instead of considering each separately we may take the limit of the characteristic (moment-generating) function, which also gives an alternative method of finding the limiting density distribution itself. To put the formulae into a suitable form for the limiting operation, denote by $F(y)$ the probability of the algebraic distance of a particle from the origin (measured positively to the right) being $\leqslant y$, and write

$$f(u) = \int_{y=-\infty}^{\infty} e^{iuy}\,dF = \sum_{\nu=-n}^{n}\gamma(n,\nu)e^{i\nu ud} = M(n,ud) = M(n,uD/n),$$

where $M(n,u)$ is given by (16). Then in the limit $f(u)\to f^*(u)$, where

$$f^*(u) = e^{-B}\left\{\cosh(B^2-u^2D^2)^{\frac{1}{2}}+B\frac{\sinh(B^2-u^2D^2)^{\frac{1}{2}}}{(B^2-u^2D^2)^{\frac{1}{2}}}\right\}, \tag{94}$$

the convergence to the limit being uniform in any finite interval for u. By general theory (**6**) the function $F^*(y)$ whose characteristic function is $f^*(u)$ is uniquely determined, and, $f^*(u)$ being continuous at $u = 0$, $F^*(y)$ is the limit of $F(y)$, at any rate at every point of continuity of $F^*(y)$. In fact, if $F^*(y)$ is defined as the limit of $F(y)$, $F^*(y)$ is known from our previous work. It has jumps of $\frac{1}{2}e^{-B}$ at $y = \pm D$, and

$$F^*(y) = 0 \quad \text{for } y < -D$$

$$= \tfrac{1}{2}e^{-B}+\frac{Be^{-B}}{2D}\int_{-D}^{y}[I_0(Y)+BI_1(Y)/Y]\,dy, \quad \text{for } -D<y<D,$$

$$= e^{-B}+\frac{Be^{-B}}{2D}\int_{-D}^{D}[I_0(Y)+BI_1(Y)/Y]\,dy, \quad \text{for } y>D, \tag{95}$$

where Y is defined in (82). To prove the connexion between $F^*(y)$ and $f^*(u)$ it is sufficient to prove either that

$$f^*(u) = \int\limits_{y=-\infty}^{\infty} e^{iuy}\,dF^* = e^{-B}\cos uD + \frac{Be^{-B}}{D}\int\limits_0^D [I_0(Y)+BI_1(Y)/Y]\cos uy\,dy, \tag{96}$$

i.e. that

$$\int\limits_0^{\frac{1}{2}\pi} [I_0(B\cos\chi)\cos\chi + I_1(B\cos\chi)]\cos(uD\sin\chi)\,d\chi = \frac{\cosh(B^2-u^2D^2)^{\frac{1}{2}}}{B} + \frac{\sinh(B^2-u^2D^2)^{\frac{1}{2}}}{(B^2-u^2D^2)^{\frac{1}{2}}} - \frac{\cos uD}{B}, \tag{97}$$

or that

$$\begin{aligned}\frac{2}{\pi}\int\limits_0^{\infty}\left[\frac{\cosh(B^2-u^2D^2)^{\frac{1}{2}}}{B} + \frac{\sinh(B^2-u^2D^2)^{\frac{1}{2}}}{(B^2-u^2D^2)^{\frac{1}{2}}} - \frac{\cos uD}{B}\right]du &= \frac{1}{D}[I_0(Y)+BI_1(Y)/Y] \quad \text{for } 0 \leqslant y < D \\ &= \frac{1}{2D}[I_0(Y)+BI_1(Y)/Y] \quad \text{at } y = D,\ Y = 0 \\ &= 0 \quad \text{for } y > D,\end{aligned} \tag{98}$$

each of which is the Fourier transform of the other. The first of these may be deduced from Sonine's second finite integral (ref. **5**, p. 376, section 12.13), and the second from Sonine's discontinuous integral (ref. **5**, p. 415, section 13.47).

This section may be closed by a final remark about the continuous distribution (89). We have taken the limit when $n \to \infty$, $\tau \to 0$, with T $(= n\tau)$ fixed. As T varies for the continuous distribution it will be the constant A $(= \lim \tau/(1-c))$ that will be independent of T. In the limit $c \to 1$, and $B = T/2A$. If we make this substitution, the formula for $\overline{y^2}$ in (74) becomes (2) or (22), and for large T,

$$\overline{y^2} \sim 2Av^2T. \tag{99}$$

It is easily proved that

$$\frac{B}{2D}e^{-B}\left[I_0(Y)+B\frac{I_1(Y)}{Y}\right] \sim (2\pi\overline{y^2})^{-\frac{1}{2}}\exp[-y^2/(2\overline{y^2})], \tag{100}$$

with the above value for $\overline{y^2}$, so the distribution tends to the Gaussian; and it is easy to verify directly that the characteristic function for the variable $y/(\overline{y^2})^{\frac{1}{2}}$ tends to $\exp(-\frac{1}{2}u^2)$.

8. The governing differential equation as the limit of the difference equation. Solution of the telegraph equation, without leakage, for an instantaneous plane source

Let us now consider the matter in a rather more physical fashion. (To begin with, change the notation slightly, and write t instead of T for the total time of the motion.) If there are $2N$ particles altogether, we see that they spread out symmetrically from the origin, N to the right and N to the left, with a 'wave-velocity' v, and that those to the right heap up at $y = vt$, where there are $Ne^{-t/2A}$, that there are none at $y > vt$, and that in $0 < y < vt$ they are spread out with a line density $N\sigma(t, y)$, where

$$\sigma(t,y) = \frac{e^{-t/2A}}{2Av}\left[I_0(Y)+\frac{t}{2A}\frac{I_1(Y)}{Y}\right], \tag{101}$$

and
$$Y = \frac{(v^2t^2-y^2)^{\frac{1}{2}}}{2Av}. \tag{102}$$

We may further consider the process involved by deriving its governing differential equation as the limit of the difference equation (12) for $\gamma(n, \nu)$. The differential of time, dt, will correspond to the interval τ, and, as previously explained, the differential of distance, dy, to $2d$ or $2v\tau = 2v\,dt$; c will be replaced by $1-dt/A$ and $p\ (= \frac{1}{2}(1+c))$ by $1-dt/2A$, so, apart from the discontinuity at $y = vt$, σ will satisfy the limit of the equation

$$\sigma(t+dt,y) = \left(1-\frac{dt}{2A}\right)[\sigma(t,y-\tfrac{1}{2}dy)+\sigma(t,y+\tfrac{1}{2}dy)]-\left(1-\frac{dt}{A}\right)\sigma(t-dt,y). \tag{103}$$

If we neglect terms of a higher order than $(dt)^2$, this becomes

$$\sigma+\frac{\partial\sigma}{\partial t}dt+\frac{1}{2}\frac{\partial^2\sigma}{\partial t^2}(dt)^2 = \left(1-\frac{dt}{2A}\right)\left[2\sigma+\frac{1}{4}\frac{\partial^2\sigma}{\partial y^2}(dy)^2\right]-\left(1-\frac{dt}{A}\right)\left[\sigma-\frac{\partial\sigma}{\partial t}dt+\frac{1}{2}\frac{\partial^2\sigma}{\partial t^2}(dt)^2\right], \tag{104}$$

i.e.
$$\left(\frac{\partial^2\sigma}{\partial t^2}+\frac{1}{A}\frac{\partial\sigma}{\partial t}\right)(dt)^2 = \tfrac{1}{4}(dy)^2\frac{\partial^2\sigma}{\partial y^2}, \tag{105}$$

so the limiting differential equation is

$$\frac{\partial^2\sigma}{\partial t^2}+\frac{1}{A}\frac{\partial\sigma}{\partial t} = v^2\frac{\partial^2\sigma}{\partial y^2}. \tag{106}$$

This is the telegraph equation, without leakage, and with $v^2 = (LC)^{-1}$, $A = L/R$, where L, C, and R are the self-inductance, capacitance, and resistance per unit length. The equation is satisfied by the potential, and also by the charge per unit length, of a cable; the potential and charge

per unit length are proportional to one another in the theory. It is easy to verify directly that (101) is a solution of (106). It is clear from the process by which it was obtained, and it may be verified *a posteriori*, that the problem solved is that of the 'diffusion', according to the telegraph equation, of an instantaneous source at the origin; since the 'diffusion' is in one dimension, the source may be called, by analogy with the corresponding problem in the theory of the conduction of heat, an instantaneous plane source; if we have a straight cable, infinite in both directions, and at time $t = 0$ introduce a charge of 2 units at the section $y = 0$, then at any time t there is a charge $e^{-t/2A}$ at $y = vt$ and at $y = -vt$, together with a line density of charge σ given by (101) between these two sections, and no charge for $|y| > vt$; A is L/R and v is $(LC)^{-\frac{1}{2}}$ as above.

This solution is analogous to the well-known solution

$$\frac{1}{(\pi\kappa t)^{\frac{1}{2}}}\exp\left(-\frac{y^2}{4\kappa t}\right) \tag{107}$$

for the heat content per unit length in an infinite medium of thermometric conductivity κ, when an instantaneous plane source of strength 2 is introduced at $y = 0$ at time $t = 0$. (The equation of heat conduction is known, and easily shown, to be derivable by a limiting process, similar to that used above, from the difference equation (12) when the motions of a particle in two successive paths are uncorrelated, i.e. $c = 0$, $p = q = \frac{1}{2}$; the thermometric conductivity κ is $\frac{1}{2}v^2\tau = \frac{1}{2}vd$; as $n \to \infty$, $\tau \to 0$ so that $n\tau = t$ is finite, but v must be allowed to tend to infinity with $n^{\frac{1}{2}}$ (in fact $v^2 = 2\kappa n/t$) and d ($= v\tau$) $\to 0$ like $n^{-\frac{1}{2}}$, so that nd becomes infinite; the differential of the time dt, in the limiting process, is of the order of the square of the differential, dy, of the distance y. The limiting process in the case considered here requires v to remain finite, and to be fixed, instead of tending to infinity; nd remains finite, and dt and dy are the same order; the root mean-square distance from the origin in the discontinuous process is, in our case, proportional to n for large n, instead of to $n^{\frac{1}{2}}$ as for $c = 0$. For more general discussions of the passage to a continuous system from the problem of the random walk, reference may be made to refs. **7** and **8**. The case considered here appears not to have been previously discussed.)

9. A solution of the telegraph equation in various forms, and mathematical deductions

A more usual problem in the theory of the telegraph equation is that in which a semi-infinite cable (i.e. for practical purposes, a cable of length greater than vt for the greatest time t considered) has one end at $y = 0$, and initially the potential V, the charge, and the current are all zero. At time $t = 0$, V is raised to unity at $y = 0$, and it is maintained at unity at

that section. It is required to find the values of V along the cable at subsequent times. The analogous problem in heat conduction is that of finding the temperature ϑ in a semi-infinite rod, when $\vartheta = 0$ initially, and ϑ is raised to unity at $y = 0$ at time $t = 0$, and is then maintained at that value at $y = 0$. The solution of that problem is

$$\vartheta = 1-\int\limits_0^y \frac{1}{(\pi\kappa t)^{\frac{1}{2}}}\exp\left(-\frac{y^2}{4\kappa t}\right)dy = 1-\frac{2}{\sqrt{\pi}}\int\limits_0^{y/\{2(\kappa t)^{\frac{1}{2}}\}} e^{-\xi^2}\,d\xi = 1-\operatorname{erf}\frac{y}{2(\kappa t)^{\frac{1}{2}}} \tag{108}$$

and the required solution of the telegraph equation is related in the same way to (101) as (108) is related to (107), account being also taken of the finite concentration, $e^{-t/2A}$, at $y = vt$. The solution is

$$\left.\begin{aligned} V = 1-\int\limits_0^y \sigma(t,y)\,dy &= 1-\frac{e^{-t/2A}}{2Av}\int\limits_0^y \left[I_0(Y)+\frac{t}{2A}\frac{I_1(Y)}{Y}\right]dy \quad \text{for } y < vt \\ &= 0 \quad \text{for } y > vt, \end{aligned}\right\} \tag{109}$$

where Y is given by (102). Since $\partial\sigma/\partial y = 0$ at $y = 0$, V satisfies the same equation (106) as σ. Also, as we have already substantially pointed out in equation (90), we must have

$$\int\limits_0^{vt} \sigma(t,y)\,dy = 1-e^{-t/2A}. \tag{110}$$

[This equation reduces to (92) with $t/2A$ in place of B; its verification is therefore included in the verification of (97) (with $u = 0$ for this particular case); it is also verified quite simply by expanding the integrand in powers of $(t/2A)$ and integrating term by term; cf. ref. **5**, section 12.11, pp. 373, 374.] Hence

$$\lim_{y\to vt-0} V = e^{-t/2A} \tag{111}$$

and V has a discontinuity, with a jump $e^{-t/2A}$, at $y = vt$.

If p is used now to denote a Heaviside operator (its previous use as a probability is not required in this section), the operational representation of V is

$$V = \exp\left[-\frac{y}{v}\left(p^2+\frac{p}{A}\right)^{\frac{1}{2}}\right], \tag{112}$$

and for $y < vt$ Jeffreys (**9**) gives a solution in series

$$V = e^{-t/2A}\left[I_0(Y)+2\sum_{n=1}^{\infty}\left(\frac{W}{Y}\right)^n I_n(Y)\right], \tag{113}$$

where

$$W = \frac{vt-y}{2Av}, \qquad \frac{W}{Y} = \left(\frac{vt-y}{vt+y}\right)^{\frac{1}{2}}. \tag{114}$$

Although it leads us away from the original purpose of this paper, it is of some interest to discuss mathematically the connexions between the different expressions of the solution. In the first place, from (112),

$$V = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \exp\left\{zt - \frac{y}{v}\left(z^2+\frac{z}{A}\right)^{\frac{1}{2}}\right\}\frac{dz}{z} \quad (c>0). \tag{115}$$

$V = 0$ for $y > vt$ is an immediate deduction. For $y < vt$, the contour may, by Jordan's lemma, be transformed into $-\infty$, $(-A+, 0+)$, and may be taken to be the lower and upper sides of the negative real axis, indented by small semicircles at $-A$, and completed by a small circle at the origin, the radii of the semicircles and the circle being finally made to tend to zero. In this way, and with $z = -x = -(1+\cos\xi)/(2A)$ on the real axis, (115) becomes

$$V = 1 - \frac{e^{-t/2A}}{\pi}\int\limits_0^{\pi} \exp\left[-\frac{t\cos\xi}{2A}\right]\sin\left[\frac{y\sin\xi}{2Av}\right]\frac{\sin\xi}{1+\cos\xi}\,d\xi, \tag{116}$$

and the identity of (116) with (109) is established by integrating (79) and (80) with respect to y from 0 to y (with $B = t/(2A)$, $KB = y/(2Av)$). If in (115) we make the substitution

$$z+\frac{1}{2A} = \frac{1}{2AW}\left[\zeta+\frac{W^2}{4\zeta}\right] \tag{117}$$

in the integrand, the integral becomes

$$V = \frac{e^{-t/A}}{2\pi i}\int \exp\left(\zeta+\frac{Y^2}{4\zeta}\right)\frac{\zeta+\frac{1}{2}W}{\zeta-\frac{1}{2}W}\frac{d\zeta}{\zeta}, \tag{118}$$

and, for $y < vt$, we may take the contour of integration as

$$-\infty,\ (-\tfrac{1}{2}W+, 0+, \tfrac{1}{2}W+).$$

We now introduce the Lommel function of two variables U_n (ref. **5**, sections 16.5–16.59, pp. 537–50). From ref. **5**, section 16.58, p. 548,

$$U_n(iW, iY) = \frac{i^{n-1}}{2\pi}\int\limits_{-\infty}^{(-\frac{1}{2}W+, 0+, \frac{1}{2}W+)} \exp\left(\zeta+\frac{Y^2}{4\zeta}\right)\frac{(\frac{1}{2}W/\zeta)^n}{1-\frac{1}{4}W^2/\zeta^2}\frac{d\zeta}{\zeta}. \tag{119}$$

Since also
$$I_0(Y) = \frac{1}{2\pi i}\int\limits_{-\infty}^{(0+)} \exp\left(\zeta+\frac{Y^2}{4\zeta}\right)\frac{d\zeta}{\zeta} \tag{120}$$

(a Bessel–Schläfli integral), it follows from (118) that, for $y < vt$,

$$V = e^{-t/2A}[2U_0(iW, iY) - 2iU_1(iW, iY) - I_0(Y)]. \tag{121}$$

Jeffreys's series (113) follows at once from

$$U_0(iW,iY) = \sum_{m=0}^{\infty}\left(\frac{W}{Y}\right)^{2m} I_{2m}(Y), \qquad U_1(iW,iY) = i\sum_{m=0}^{\infty}\left(\frac{W}{Y}\right)^{2m+1} I_{2m+1}(Y) \tag{122}$$

(ref. **5**, section 165, p. 537).

That (109) and (113) are equivalent may also be simply seen by differentiating the right-hand side of (113) term by term with respect to y, and using the recurrence formulae for the I_n to show that the derivative is

$$-[I_0(Y)+\tfrac{1}{2}(W/Y+Y/W)I_1(Y)].$$

The following formula follows from the equivalence of (109) and (113).

$$I_0(z\sin\theta)+2\sum_{n=1}^{\infty}\tan^2\tfrac{1}{2}\theta\, I_n(z\sin\theta) = 1+z\int_0^{\theta}[I_0(z\sin\chi)\sin\chi+I_1(z\sin\chi)]\,d\chi. \tag{123}$$

[Since

$$e^z = I_0(z)+2\sum_{n=1}^{\infty} I_n(z)$$

(92) is the special case of this when $\theta = \frac{1}{2}\pi$.]

It is also of some small mathematical interest to remark that

$$e^{-t/2A}\left(\frac{vt-y}{vt+y}\right)^{\frac{1}{2}n} I_n(Y) \tag{124}$$

is the solution for $y < vt$ when the end condition is $V = e^{-t/2A}I_n(t/2A)$, whether n be integral or not. The operational representation of this solution is

$$\frac{1}{(2A)^n}\left(\frac{p}{p+1/A}\right)^{\frac{1}{2}}\left[p+\frac{1}{2A}+\left(p^2+\frac{p}{A}\right)^{\frac{1}{2}}\right]^{-n}\exp\left[-\frac{y}{v}\left(p^2+\frac{p}{A}\right)^{\frac{1}{2}}\right]; \tag{125}$$

that (124) is the interpretation of (125) may be established by proving either that

$$\tan^n\tfrac{1}{2}\theta\, I_n(x\sin\theta) = \frac{1}{\pi}\int_0^{\pi}\exp(x\cos\xi)\cos(x\cos\theta\sin\xi+n\xi)\,d\xi-$$

$$-\frac{\sin n\pi}{\pi}\int_0^{\pi}\exp(-x\cosh\xi+x\cos\theta\sinh\xi-n\xi)\,d\xi, \tag{126}$$

or that

$$\left.\begin{aligned}&\int_0^{\infty}\exp[-Z(\cosh\xi-1)]I_n(z\sinh\xi)\tanh^n\tfrac{1}{2}\xi\sinh\xi\,d\xi\\ &\quad=\int_0^{\infty}e^{-uZ}I_n[z(u^2+2u)^{\frac{1}{2}}]\left[\frac{u}{(u^2+2u)^{\frac{1}{2}}}\right]^n du\\ &\quad=\frac{\exp[Z-(Z^2-z^2)^{\frac{1}{2}}]}{[Z+(Z^2-z^2)^{\frac{1}{2}}]^n}\,\frac{1}{(Z^2-z^2)^{\frac{1}{2}}}\quad (Z>z>0)\end{aligned}\right\} \tag{127}$$

of which one is deducible from the Mellin transformation of the other; (126) may be deduced from the formula (5) on page 177, section 6.21, of ref. **5** (due to Sonine) for $\mathrm{re}[x(1-\cos\theta)] > 0$; (127) may be deduced independently, for $\mathrm{re}(Z \pm z) > 0$, by integrating term by term the expansion

$$\frac{I_n[z(u^2+2u)^{\frac{1}{2}}]}{z^2(u^2+2u)^{\frac{1}{2}n}} = \sum_{m=0}^{\infty} \frac{z^{m-n}}{m!\,u^n} I_{n+m}(uz) \tag{128}$$

after multiplication by $e^{-uZ}u^n$, and using the formula

$$\int_0^\infty e^{-au} I_n(bu)\, du = \frac{[a-(a^2-b^2)^{\frac{1}{2}}]^n}{b^n(a^2-b^2)^{\frac{1}{2}}} \tag{129}$$

(ref. **5**, p. 386); the expansion (128) is a slightly altered form of Lommel's expansion (ref. **5**, section 5.22, p. 140).

10. Further developments. The problem with leakage included

Certain generalizations of the preceding work are suggested. One of the most immediate and obvious, which is not uninteresting, is to generalize the work to include leakage in the final form of the telegraph equation. The whole discussion has started, of course, as a generalization of the classical problem of 'the drunkard's walk', and the further generalization required is this: to find the probability, after he has had time for a fixed number of steps, that a drunken man will be found a certain number of steps forward from his initial position if he manages to keep moving in a straight line (if he does not disappear) and there is a probability p at each step of forward motion and a probability q of backward motion, but also a non-zero probability, $1-p-q$, after each step that he will, in fact, fall down a man-hole and apparently disappear altogether. In terms of the motion of a large number of particles, with which we started, we must dispense with the condition $p+q = 1$, and allow a non-zero probability $1-p-q$ that a particle will escape from the system at the end of a partial path. We may define $\alpha(n,\nu)$, $\beta(n,\nu)$, $\gamma(n,\nu)$ as before, but we must now add that they are to be taken at time $n\tau-0$ (i.e. before the disappearance of a certain fraction of the total number of particles at time $n\tau$). We still write $c = p-q$, so that c is the correlation coefficient between the directions in two successive partial paths for those particles that do not escape from the system; we also write $r = p/q$ as before. Then many of our equations are unaltered—e.g. (4)–(10) inclusive. Equation (11) is replaced by

$$\alpha(n,\nu+1)+\beta(n,\nu-1) = (p+q)\gamma(n-1,\nu), \tag{130}$$

and (12) and (15) by

$$\gamma(n+1,\nu) = p[\gamma(n,\nu-1)+\gamma(n,\nu+1)]-(p^2-q^2)\gamma(n-1,\nu), \tag{131}$$

and
$$M(n+1,u) = 2p\cos uM(n,u)-(p^2-q^2)M(n-1,u), \tag{132}$$

respectively; (14) is slightly altered to

$$M(1) = \cos u, \qquad M(2) = q+p\cos 2u = p+q-2p\sin^2 u,$$

and the solution for $M(n,u)$ is still of the form (16), with a_1 and a_2 still given by (18), but with

$$k_1 = \frac{1}{2(p+q)}\left[1+\frac{\cos u}{(1-r^2\sin^2 u)^{\frac{1}{2}}}\right], \qquad k_2 = \frac{1}{2(p+q)}\left[1-\frac{\cos u}{(1-r^2\sin^2 u)^{\frac{1}{2}}}\right]. \tag{133}$$

If we express the previous values of $M(n,u)$ and $\gamma(n,\nu)$ in terms of $r = p/q$, in addition to n and u, and to n and ν, respectively, then the new values are the same as the old values multiplied by $(p+q)^{n-1}$. We now have, for example,

$$\sum_{\nu=-n}^{n} \gamma(n,\nu) = (p+q)^{n-1}, \tag{134}$$

and
$$\sum_{\nu=-n}^{n} \nu^2\gamma(n,\nu) = (p+q)^{n-1}\left[nr-\tfrac{1}{2}(r^2-1)\left\{1-\left(\frac{r-1}{r+1}\right)^n\right\}\right], \tag{135}$$

corresponding with (20).

If we put
$$\frac{\tau}{1-c} = A, \tag{136}$$

as before, and also
$$\frac{\tau}{1-(p+q)} = G, \tag{137}$$

and let $n \to \infty$, $\tau \to 0$, with $n\tau = t$, as before, then in the limit the mean-square distance from the origin is given by

$$\overline{y^2} = v^2 e^{-t/G}\left[\frac{2t}{1/A-1/G}-\frac{2}{(1/A-1/G)^2}\{1-e^{-t(1/A-1/G)}\}\right], \tag{138}$$

corresponding with (22). $1/G$ is the proportion of the charge that leaks away per unit time.

If we now form a differential equation from (131) in the same way as (106) was formed from (12), the result is

$$\frac{\partial^2\sigma}{\partial t^2}+\left(\frac{1}{A}+\frac{1}{G}\right)\frac{\partial\sigma}{\partial t}+\frac{\sigma}{AG} = v^2\frac{\partial^2\sigma}{\partial y^2}, \tag{139}$$

which is the telegraph equation with leakage, with $v^2 = (LC)^{-1}$ and $A = L/R$ as before, and with $G = C/S$, where $1/S$ is the leakage resistance per unit length. Since the charge per unit length is CV, where V is the potential, and the leakage current per unit length is SV, this is the result we should expect for G.

Equation (139) may be written

$$\frac{\partial^2\sigma'}{\partial t^2}+\left(\frac{1}{A}-\frac{1}{G}\right)\frac{\partial\sigma'}{\partial t}=v^2\frac{\partial^2\sigma'}{\partial y^2},\qquad \sigma=e^{-t/G}\sigma', \tag{140}$$

and we may easily convince ourselves that not only (138) and (139), but all the limiting results, are obtained from the previous results by replacing A^{-1} by $A^{-1}-G^{-1}$ and multiplying by $e^{-t/G}$. Thus (101) is replaced by

$$\sigma(t,y)=\frac{1}{2v}\left(\frac{1}{A}-\frac{1}{G}\right)\exp\left[-\frac{t}{2}\left(\frac{1}{A}+\frac{1}{G}\right)\right]\left\{I_0(Y)+\frac{t}{2}\left(\frac{1}{A}-\frac{1}{G}\right)\frac{I_1(Y)}{Y}\right\}, \tag{141}$$

where Y is now given by

$$Y^2=\frac{1}{4}\left(\frac{1}{A}-\frac{1}{G}\right)^2\left(t^2-\frac{y^2}{v^2}\right) \tag{142}$$

in place of (102). (It should be added that, in order that q should be positive, $1/A-1/G$ must be positive.) The finite concentration at $y=\pm vt$ is now $\exp[-\frac{1}{2}t(1/A+1/G)]$.

The result corresponding to (109) is the solution of the problem when $V=e^{-t/G}$ at $y=0$ for $t\geqslant 0$, and is obtained by replacing A^{-1} by $A^{-1}-G^{-1}$ and multiplying by $e^{-t/G}$ as before. To obtain the solution of (139) with the end condition $V=1$ at $y=0$ for $t\geqslant 0$, we must start from the solution of (106) with the end condition $V=e^{t/G}$ at $y=0$ for $t\geqslant 0$, and then replace A^{-1} by $A^{-1}-G^{-1}$ and multiply by $e^{-t/G}$, as before. The required solution of (106) may be found from that given in (109); if $V_1(t,y)$ denotes that solution for $y<vt$, the solution of (106) with the end condition $V=f(t)$ at $y=0$ for $t\geqslant 0$ is

$$V=f(0)V_1(t,y)+\int\limits_{y/v}^{t}f'(t-\xi)V_1(\xi,y)\,d\xi, \tag{143}$$

where $f'(t)$ is df/dt. It may be added that, with leakage present, the solution for $V=1$ at $y=0$ for $t\geqslant 0$ may also be expressed in terms of Lommel's function U_n of two variables: it is (for $y<vt$)

$$V=\exp\left\{-\frac{t}{2}\left(\frac{1}{A}+\frac{1}{G}\right)\right\}[U_0(iW_1,iY)+U_0(iW_2,iY)-iU_1(iW_1,iY)-$$
$$-iU_1(iW_2,iY)-I_0(Y)], \tag{144}$$

where $$W_1=\frac{1}{2}\left(\frac{1}{A^{\frac{1}{2}}}+\frac{1}{G^{\frac{1}{2}}}\right)^2\left(t-\frac{y}{v}\right),\qquad W_2=\frac{1}{2}\left(\frac{1}{A^{\frac{1}{2}}}-\frac{1}{G^{\frac{1}{2}}}\right)^2\left(t-\frac{y}{v}\right). \tag{145}$$

This solution holds whether $A^{-1}-G^{-1}$ is positive or negative; Y must be taken as the positive square root of (142). Jeffreys's series for this case is immediately deducible, but I have not pursued the matter further.

Other generalizations lead to higher order difference and differential equations. I have, for example, briefly considered the case when, in

addition to non-zero probabilities that a particle will move forward or backward with a velocity v, there is also a non-zero probability that it will come to rest (the swaying drunkard), together with a non-zero probability that it will start again from rest with velocity v. This leads to a third-order hyperbolic differential equation in the limit, containing $\partial^3\sigma/\partial t^3$, $\partial^2\sigma/\partial t^2$, $\partial\sigma/\partial t$, $\partial^3\sigma/\partial y^2\partial t$, $\partial^2\sigma/\partial y^2$, the highest terms being

$$\partial^3\sigma/\partial t^3 - v^2\, \partial^3\sigma/\partial y^2\partial t.$$

Other problems arise by introducing non-zero partial correlations between the motions in non-consecutive intervals.

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REFERENCES

1. G. I. Taylor, 'Diffusion by continuous movements', *Proc. London Math. Soc.* (2) **20** (1921/2), 196–212.
2. J. Wishart and H. O. Hirschfeld, 'A theorem concerning the distribution of joins between line segments', *Journ. London Math. Soc.* **11** (1936), 227–35.
3. E. W. Montroll, 'On the theory of Markoff chains', *Annals of Math. Statistics*, **18** (1947), 18–36.
4. Lord Rayleigh, 'On the problem of random vibrations, and of random flights in one, two, or three dimensions', *Scientific Papers*, vol. 6, pp. 604–26, particularly pp. 607–9 (from *Phil. Mag.* **37** (1919), 321–47).
5. G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1922).
6. H. Cramer, *Random Variables and Probability Distributions* (Cambridge Tracts in Mathematics and Mathematical Physics, No. 36 (1937)), pp. 28–31.
7. S. Chandrasekhar, 'Stochastic problems in physics and astronomy', *Reviews of Modern Physics*, **15** (1943), 1–89.
8. J. E. Moyal, 'Stochastic processes and statistical physics', *Journ. Roy. Statistical Soc.* (B), **11** (1949), 150–210. (See p. 195 et seq.).
9. H. Jeffreys, *Operational Methods in Mathematical Physics* (2nd edition) (Cambridge Tracts in Mathematics and Mathematical Physics, No. 23 (1931)), pp. 104–8.

Note added, 26 October 1950

I am indebted to Professor Kampé de Fériet for pointing out to me the known connexion of the formula (28) with the result in equation (84), and to Professor Erdélyi for further discussion and help in this connexion. If $F(a,b;c;z)$ is the hypergeometric function, defined by

$$F(a,b;c;z) = 1+\frac{a\,.\,b}{1\,.\,c}z+\frac{a(a+1)b(b+1)}{2!\,c(c+1)}z^2+\dots \quad \text{for } |z|<1,$$

(28) may be written

$$\gamma(n,\nu) = \frac{r^{n-3}(r-1)}{(r+1)^{n-1}} F\left(-\frac{n-\nu}{2}+1,\ -\frac{n+\nu}{2}+1;\ 1;\ \frac{1}{r^2}\right)+$$
$$+\frac{nr^{n-3}}{2(r+1)^{n-1}} F\left(-\frac{n-\nu}{2}+1;\ -\frac{n+\nu}{2}+1;\ 2;\ \frac{1}{r^2}\right)$$

for $\nu < n$. It is, in fact, a known result that the solution of a difference equation such as (12) may be expressed in terms of hypergeometric functions. In accordance with the notation of section 6, we now put $r = n/B$, $\nu = Kn$, and let $n \to \infty$. From a result given in ref. **5**, section 5.7, p. 154, it is known that

$$\lim_{\substack{\lambda\to\infty \\ \mu\to\infty}} \frac{1}{m!} F\left(\lambda, \mu; m+1; \frac{z^2}{4\lambda\mu}\right) = I_m(Y)/(\tfrac{1}{2}Y)^m,$$

where $Y = \lim z$. The limiting form of $\gamma(n,\nu)$ for $\nu < n$, given in (84), follows. (The limit given in (86) follows at once from (27), since

$$p = r/(r+1) = (1+B/n)^{-1}.)$$

SOME ASPECTS OF RAYLEIGH'S PROBLEM FOR A COMPRESSIBLE FLUID

By L. HOWARTH (*Department of Mathematics, The University, Bristol*)

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SUMMARY

Rayleigh's problem, in which a plane is started to move parallel to itself through viscous fluid, is considered for the case when the fluid is compressible. An approximate expression for the pressure-time variation at the plane is obtained and the initial motion is considered. It is shown that the initial pressure variations determined, as here, from the classical equations are so rapid as to require reconsideration in the light of molecular phenomena, though there is little doubt that the entire pressure changes are limited to an interval, short by ordinary standards, and governed essentially by the time of relaxation. This conclusion in itself is thought to be of some interest in the theory of viscous compressible flow, and its implications in relation to the steady flow along a semi-infinite plane are considered.

1. Introduction

The motion set up when an infinite plane, immersed in viscous incompressible fluid, is set moving instantaneously from rest in a direction parallel to itself (and thereafter maintained in uniform motion) is one of the standard problems of incompressible flow. The motion is everywhere parallel to the imposed motion of the plane, and the velocity is given by

$$U(1-\operatorname{erf} y/2(\nu t)^{\frac{1}{2}}),$$

where U is the velocity of the plane, t is the time, y is the perpendicular distance from the plane, and ν is the kinematic viscosity.

In the present paper we shall consider the corresponding problem when the fluid is compressible, and we shall, in fact, limit attention to a perfect gas. Compressibility, of course, considerably complicates the problem since the temperature variations, which occur through dissipation, induce density variations which in turn give rise to a component of flow at right angles to the plane. We shall limit attention to two aspects of the problem:

(i) an approximate solution for the pressure–time variation at the plane when the Mach number is small and the Prandtl number $\sigma = \frac{3}{4}$;

(ii) a representation of the initial motion for the full equations when temperature variations of the coefficient of viscosity are neglected.

The approximate solution referred to in (i) is obtained by linearizing the equations in all save the dissipation terms, but is an improvement on the analysis of Lagerstrom, Cole, and Trilling (**1**) (hereafter referred to as L.C.T.), who neglected conductivity and did no more than obtain a repeated integral in complex form for a general value of the dissipation.

We shall allow for conductivity in the special case when the Prandtl number is $\frac{3}{4}$ and shall derive, at the plane, the particular solution associated with the actual dissipation as a single finite real integral, which will be tabulated.

It should be said at the outset that, for air under standard conditions, the solution derived below on the basis of the classical equations gives significant pressure variations in times of the order of 5×10^{-10} sec. Modifications of our results in the early stages must therefore be expected in the light of molecular considerations. Further discussion of this feature of the solution is reserved until section 5.

2. Equations of motion

With Ox measured parallel to the direction of motion and Oy perpendicular to the plate the equations of flow are in the usual notation*

$$\rho\frac{Du}{Dt} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{1}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y}+\frac{4}{3}\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right), \tag{2}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial y}(\rho v) = 0, \tag{3}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t}+v\frac{\partial}{\partial y}. \tag{4}$$

To these must be added the equation of state

$$p = R\rho T \tag{5}$$

and the energy equation

$$\rho J c_p\frac{DT}{Dt}-\frac{Dp}{Dt} = J\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left[\left(\frac{\partial u}{\partial y}\right)^2+\frac{4}{3}\left(\frac{\partial v}{\partial y}\right)^2\right]. \tag{6}$$

From (3), a function ψ exists such that

$$\frac{\rho}{\rho_s} = \frac{\partial\psi}{\partial y}, \qquad \frac{\rho}{\rho_s}v = -\frac{\partial\psi}{\partial t}, \tag{7}$$

where here and elsewhere the suffix s is used to denote some standard condition of the gas which will, in fact, be the undisturbed state.

We may, then, in von Mises's fashion, transform to x and ψ as independent variables as indeed Illingworth (**2**) has done in the boundary layer approximation to the present problem. Then we have

$$\left(\frac{\partial}{\partial t}\right)_y = \left(\frac{\partial}{\partial t}\right)_\psi - \frac{\rho}{\rho_s}v\left(\frac{\partial}{\partial \psi}\right)_t, \tag{8}$$

$$\left(\frac{\partial}{\partial y}\right)_t = \frac{\rho}{\rho_s}\left(\frac{\partial}{\partial \psi}\right)_t, \tag{9}$$

* All x-derivatives are clearly zero.

so that
$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{\psi}. \tag{10}$$

Finally putting

$$\psi = \frac{\nu_s}{a_s}\psi', \qquad t = \frac{\nu_s}{a_s^2}t', \qquad u = Uu', \qquad v = a_s v'$$
$$p = p_s p', \qquad \rho = \rho_s \rho', \qquad T = T_s T' \tag{11}$$

and then dropping the dashes we have

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial \psi}\left(\mu\rho\frac{\partial u}{\partial \psi}\right), \tag{12}$$

$$\frac{\partial v}{\partial t} = \frac{4}{3}\frac{\partial}{\partial \psi}\left(\mu\rho\frac{\partial v}{\partial \psi}\right) - \frac{1}{\gamma}\frac{\partial p}{\partial \psi}, \tag{13}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = \frac{\partial v}{\partial \psi}, \tag{14}$$

$$p = \rho T, \tag{15}$$

$$\frac{\partial T}{\partial t} - \frac{\gamma-1}{\gamma}\frac{1}{\rho}\frac{\partial p}{\partial t} = (\gamma-1)\mu\rho\left[M^2\left(\frac{\partial u}{\partial \psi}\right)^2 + \frac{4}{3}\left(\frac{\partial v}{\partial \psi}\right)^2\right] + \frac{1}{\sigma}\frac{\partial}{\partial \psi}\left(\mu\rho\frac{\partial T}{\partial \psi}\right), \tag{16}$$

where σ has been written for $\mu c_p/k$, γ is the ratio of the specific heats, and $M = U/a_s$.

An alternative form of (16) obtained by using (14) is

$$\frac{1}{\gamma}\frac{\partial T}{\partial t} + \frac{\gamma-1}{\gamma}p\frac{\partial v}{\partial \psi} = (\gamma-1)\mu\rho\left[M^2\left(\frac{\partial u}{\partial \psi}\right)^2 + \frac{4}{3}\left(\frac{\partial v}{\partial \psi}\right)^2\right] + \frac{1}{\sigma}\frac{\partial}{\partial \psi}\left(\mu\rho\frac{\partial T}{\partial \psi}\right). \tag{17}$$

The boundary conditions are, assuming the plate to be thermally insulating, and taking conditions in the gas at rest as standard,

$$\left.\begin{array}{l} u = 1, \quad v = 0, \quad \dfrac{\partial T}{\partial \psi} = 0 \quad \text{at } \psi = 0, \\ u = 0, \quad v = 0, \quad p = 1, \quad \rho = 1, \\ \qquad T = 1 \text{ at } t = 0 \text{ and as } \psi \to \infty. \end{array}\right\} \tag{18}$$

3. The linearized solution for small M^2 when $\sigma = \frac{3}{4}$

If M^2 is small, the variations of pressure, density, and temperature are small and so is the velocity component normal to the plate. Therefore, if we write

$$p = 1+p', \quad \rho = 1+\rho', \quad T = 1+T', \quad \mu = 1+\mu', \quad v = 1+v', \tag{19}$$

the squares and products of the dashed quantities may be neglected for a

first approximation and we have from (13)–(16), again dropping the dashes,

$$\frac{\partial v}{\partial t} = \frac{4}{3}\frac{\partial^2 v}{\partial\psi^2} - \frac{1}{\gamma}\frac{\partial p}{\partial\psi}, \tag{20}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial v}{\partial\psi} = 0, \tag{21}$$

$$\frac{\partial T}{\partial t} - \frac{\gamma-1}{\gamma}\frac{\partial p}{\partial t} = \Phi + \frac{1}{\sigma}\frac{\partial^2 T}{\partial\psi^2}, \tag{22}$$

$$p = \rho + T, \tag{23}$$

where Φ is the dissipation function which, to a first approximation, is given by

$$\Phi = (\gamma-1)M^2\left(\frac{\partial u}{\partial\psi}\right)^2. \tag{24}$$

The velocity u itself is $O(1)$, but since it affects (20)–(22) only through Φ which contains the factor M^2 it is sufficient to determine the leading term only in u, and this is clearly given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial\psi^2}, \tag{25}$$

so that

$$u = 1 - \operatorname{erf}(\psi/2t^{\frac{1}{2}}), \tag{26}$$

and therefore

$$\Phi = \frac{(\gamma-1)M^2}{\pi t}e^{-\psi^2/2t}, \tag{27}$$

to the order required.

The boundary conditions on v, p, ρ, T are now

$$\left.\begin{array}{lll} \text{(i)} & v = 0, \quad \dfrac{\partial T}{\partial\psi} = 0 & \text{when } \psi = 0, \\ \text{(ii)} & v,\ p,\ T,\ \rho \to 0 & \text{as } \psi \to \infty. \end{array}\right\} \tag{28}$$

Now it follows from (21) and (23) that

$$\frac{\partial T}{\partial t} = \frac{\partial p}{\partial t} + \frac{\partial v}{\partial\psi}, \tag{29}$$

and from (22) that

$$\left(\frac{\partial}{\partial t} - \frac{1}{\sigma}\frac{\partial^2}{\partial\psi^2}\right)\frac{\partial T}{\partial t} = \frac{\gamma-1}{\gamma}\frac{\partial^2 p}{\partial t^2} + \frac{\partial\Phi}{\partial t}. \tag{30}$$

Therefore, inserting the value of $\partial T/\partial t$ from (29) in (30) it follows that

$$\left(\frac{\partial}{\partial t} - \frac{1}{\sigma}\frac{\partial^2}{\partial\psi^2}\right)\frac{\partial p}{\partial t} + \left(\frac{\partial}{\partial t} - \frac{1}{\sigma}\frac{\partial^2}{\partial\psi^2}\right)\frac{\partial v}{\partial\psi} = \frac{\gamma-1}{\gamma}\frac{\partial^2 p}{\partial t^2} + \frac{\partial\Phi}{\partial t}. \tag{31}$$

Comparing (20) and (31), it is evident that $\sigma = \frac{3}{4}$ is a specially simple case and, since this value is so close to the observed value 0·7 for air, we shall

take advantage of the simplification offered and limit attention to this value.

Then substituting from (20) in (31) we have

$$\left(\frac{\partial}{\partial t}-\frac{4}{3}\frac{\partial^2}{\partial \psi^2}\right)\frac{\partial p}{\partial t}-\frac{1}{\gamma}\frac{\partial^2 p}{\partial \psi^2}=\frac{\gamma-1}{\gamma}\frac{\partial^2 p}{\partial t^2}+\frac{\partial \Phi}{\partial t}, \tag{32}$$

that is,

$$\frac{4}{3}\gamma\frac{\partial^3 p}{\partial \psi^2\,\partial t}+\frac{\partial^2 p}{\partial \psi^2}-\frac{\partial^2 p}{\partial t^2}=-\gamma\frac{\partial \Phi}{\partial t}. \tag{33}$$

This equation is exactly of the form obtained by L.C.T. (**1**) who neglected conductivity altogether. L.C.T. (**1**) considered the solution of (33) for a general value of the right-hand side only and used influence functions to express the solution as repeated integrals. We, on the other hand, shall make use of the known form of Φ and shall consider the solution of that specific equation.

The boundary conditions on p may be determined as follows. Firstly, since $v = 0$ when $\psi = 0$, it follows from (20) that, at $\psi = 0$,

$$\frac{4}{3}\frac{\partial^2 v}{\partial \psi^2}=\frac{1}{\gamma}\frac{\partial p}{\partial \psi}. \tag{34}$$

Also, from (29) by differentiation

$$\frac{\partial^2 T}{\partial t\partial \psi}=\frac{\partial^2 p}{\partial t\partial \psi}+\frac{\partial^2 v}{\partial \psi^2}. \tag{35}$$

Now, since we are considering a thermally insulating boundary, $\partial T/\partial \psi = 0$ when $\psi = 0$ and, since this is true for all time, $\partial^2 T/\partial t\partial \psi = 0$. Therefore, from (34) and (35),

$$\frac{\partial^2 p}{\partial t\partial \psi}+\frac{3}{4\gamma}\frac{\partial p}{\partial \psi}=0 \tag{36}$$

when $\psi = 0$ for all t.

That is,

$$\left(\frac{\partial p}{\partial \psi}\right)_{\psi=0}=Ae^{-3t/4\gamma}. \tag{37}$$

But $(\partial p/\partial \psi)_{\psi=0} \to 0$ when $t \to 0$;* $(\partial p/\partial \psi)_{\psi=0}$ must therefore vanish for all time. Hence our boundary conditions on p are

$$p=\frac{\partial p}{\partial t}=0 \quad \text{when } t=0\ (\psi>0) \text{ and when } \psi\to\infty,$$

$$\frac{\partial p}{\partial \psi}=0 \quad \text{when } \psi=0\ (t>0).$$

Hence, with a convenient change of variables,

$$\tau=3t/4\gamma, \qquad \chi=3\psi/4\gamma, \tag{38}$$

* See, for example, the initial motion in section 4.

we require to solve the equation

$$\frac{\partial^3 p}{\partial\chi^2\partial\tau}+\frac{\partial^2 p}{\partial\chi^2}-\frac{\partial^2 p}{\partial\tau^2}=\frac{\beta}{2}\frac{\partial}{\partial\tau}\left(\frac{1}{\tau}e^{-4\gamma\chi^2/3\tau}\right), \tag{39}$$

where $\beta=-2\gamma(\gamma-1)M^2/\pi$,

with the boundary conditions

$$\left.\begin{aligned} p=\frac{\partial p}{\partial\tau}=0 \quad &\text{when } \tau=0\ (\chi>0) \text{ and as } \chi\to\infty \\ \frac{\partial p}{\partial\chi}=0 \quad &\text{when } \chi=0\ (\tau>0). \end{aligned}\right\} \tag{40}$$

This is a problem admirably suited to the application of Heaviside's method since the right-hand side can be expressed operationally in terms of Bessel functions.* To avoid confusion we shall henceforward write P for the excess pressure and p for Heaviside's operator $\partial/\partial\tau$. Then, since

$$pK_0(p^{\frac{1}{2}}\chi/h^{\frac{1}{2}})=\frac{1}{2\tau}e^{-\chi^2/4h\tau} \tag{41}$$

(where K_0 is Bessel's function of the second kind of order zero) we may put the right-hand side of (39) in the form

$$\beta p^2K_0[\chi(8\gamma p/3)^{\frac{1}{2}}]. \tag{42}$$

Hence the operational form of (39) is

$$\frac{\partial^2 P}{\partial\chi^2}-m^2P=\beta m^2K_0[p^{\frac{1}{2}}\alpha\chi], \tag{43}$$

where $m^2=p^2/(p+1)$ and $\alpha=(8\gamma/3)^{\frac{1}{2}}$. (44)

Variation of parameters leads to the operational solution

$$P=-\beta m\left[\cosh m\chi\int\limits_{\chi}^{\infty}e^{-m\zeta}K_0(p^{\frac{1}{2}}\alpha\zeta)\,d\zeta+e^{-m\chi}\int\limits_{0}^{\chi}\cosh m\zeta K_0(p^{\frac{1}{2}}\alpha\zeta)\,d\zeta\right], \tag{45}$$

and, in particular, the excess pressure P_W at the wall is

$$P_W=-\beta m\int\limits_0^\infty e^{-m\zeta}K_0(p^{\frac{1}{2}}\alpha\zeta)\,d\zeta \tag{46}$$

$$=-\beta n\int\limits_0^\infty e^{-n\mu}K_0(\mu)\,d\mu, \tag{47}$$

where $n=m/(\alpha p)^{\frac{1}{2}}=[p/\alpha(1+p)]^{\frac{1}{2}}$. (48)

Hence (see Watson's *Bessel Functions*, p. 388),

$$P_W=-(\beta n\cos^{-1}n)/(1-n^2)^{\frac{1}{2}}. \tag{49}$$

* For an account of this method and its relation to the Laplace transform the reader is referred to Jeffreys and Jeffreys (**3**).

The expressions (45) and (49) have then to be interpreted by means of Bromwich's integral, so that we have, from (45),

$$P = -\frac{\beta}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{e^{\lambda\tau}}{(1+\lambda)^{\frac{1}{2}}}\left\{\cosh\left[\frac{\lambda\chi}{(1+\lambda)^{\frac{1}{2}}}\right]\int\limits_{\chi}^{\infty} e^{-\lambda\zeta/(1+\lambda)^{\frac{1}{2}}} K_0(\lambda^{\frac{1}{2}}\alpha\zeta)\, d\zeta + \right.$$

$$\left. + e^{-\lambda\chi/(1+\lambda)^{\frac{1}{2}}} \int\limits_{0}^{\chi} \cosh\left[\frac{\lambda\zeta}{(1+\lambda)^{\frac{1}{2}}}\right] K_0(\lambda^{\frac{1}{2}}\alpha\zeta)\, d\zeta\right\} d\lambda. \quad (50)$$

As L.C.T. (**1**) have shown, the exponential essential singularity at $\lambda = -1$ makes further progress difficult except by way of asymptotic expressions which are clearly insufficient when, as here, the result has to be integrated again. As they also remark, the functions represented by integrals of the type

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} e^{\lambda\tau-\lambda\chi/(1+\lambda)^{\frac{1}{2}}}\frac{\phi(\lambda)}{\lambda}\, d\lambda, \quad (51)$$

where $\phi(\lambda)$ is reasonably well behaved, will have to be studied further before the determination of conditions throughout the field of flow can be completed.

Fortunately the form given in (49) when $\chi = 0$, similarly interpreted, has only a logarithmic singularity at $\lambda = -1$ and can be dealt with satisfactorily. For, in virtue of (48) and (49), we can write

$$P_W = -\frac{\beta p^{\frac{1}{2}}}{[(\alpha^2-1)p+\alpha^2]^{\frac{1}{2}}}\tan^{-1}\left[\frac{\alpha^2+(\alpha^2-1)p}{p}\right]^{\frac{1}{2}}, \quad (52)$$

and by Bromwich's integral the interpretation is

$$P_W = -\frac{\beta}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{e^{\lambda\tau}}{\lambda^{\frac{1}{2}}[(\alpha^2-1)\lambda+\alpha^2]^{\frac{1}{2}}}\tan^{-1}\left[\frac{\alpha^2+(\alpha^2-1)\lambda}{\lambda}\right]^{\frac{1}{2}} d\lambda. \quad (53)$$

The integrand has branch points at $\lambda = 0$ and $\lambda = -\alpha^2/(\alpha^2-1) = -b$ say, and $\lambda = -1$. The last named is in fact a logarithmic singularity; for

$$\tan^{-1}z = \frac{1}{2i}\log\frac{1+iz}{1-iz}$$

and has branch points (like $\log z$ at $z = 0$) when $z = \pm i$ (i.e. $z^2 = -1$) which corresponds in our case to $\lambda = -1$.

The usual arguments suffice to show that the contour of integration can be modified to the two sides of the negative axis indented at the three branch points. The integrals about each of the parts of the three indentations tend to zero with the radius of the indentation. Hence we are left with the integrals from the various straight portions.

The three branch points make the evaluation tedious without being in any way novel and the analysis will not be reproduced here. The results obtained are as follows:

(1) The contribution to the integral from the two sides between $-b$ and $-\infty$ and between -1 and $-b$ are found to be of the same magnitude and opposite in sign and therefore cancel.

(2) The contributions between 0 and 1 give

$$P_W = \frac{-\beta}{2(\alpha^2-1)^{\frac{1}{2}}}\int_0^1 \frac{e^{-\mu\tau}}{\mu^{\frac{1}{2}}(b-\mu)^{\frac{1}{2}}}\,d\mu = \frac{-\beta}{(\alpha^2-1)^{\frac{1}{2}}}\int_0^{\sin^{-1}b^{-\frac{1}{2}}} e^{-b\tau\sin^2\theta}\,d\theta. \tag{54}$$

The pressure therefore rises instantaneously at time zero by an amount

$$(P_W)_{t=0} = \frac{2}{\pi}\gamma(\gamma-1)M^2\left(\frac{3}{8\gamma-3}\right)^{\frac{1}{2}}\sin^{-1}\left(\frac{8\gamma-3}{8\gamma}\right)^{\frac{1}{2}}, \tag{55}$$

which may be ascribed to the infinite initial dissipation. This result is confirmed in detail by the analysis of the next section where the manner in which the boundary conditions are satisfied is made apparent.

A few values of the excess pressure P_W (which is, it should be remembered, effectively expressed as the ratio to the undisturbed pressure p_s) are given in Table I, for $\gamma = 1{\cdot}4$, as a function of $a_s^2 t/\nu_s$ (where t is the actual time).

TABLE 1

$a_s^2 t/\nu_s$	P_W/M^2
0·0000	0·221
1·366	0·171
2·733	0·138
5·467	0·101
10·93	0·070
21·87	0·049

Asymptotically

$$P_W \sim 0{\cdot}223M^2\left(\frac{\nu_s}{a_s^2 t}\right)^{\frac{1}{2}}. \tag{56}$$

4. The initial motion for the complete equations, temperature variations of μ being neglected

We return now to the full equations of motion set out in (12)–(16). The method to be developed here would, in fact, be possible for any given temperature variation of μ, but the details are considerably simpler when μ is constant and we shall restrict our discussion to that case.

The solution of the corresponding problem for incompressible flow suggests the use of a new variable

$$\xi = \psi/t^{\frac{1}{2}}, \tag{57}$$

and we shall for convenience associate with it as the second independent variable

$$\eta = t^{\frac{1}{2}}. \tag{58}$$

Then
$$\frac{\partial}{\partial t} = \frac{1}{2\eta^2}\left(\eta\frac{\partial}{\partial\eta} - \xi\frac{\partial}{\partial\xi}\right) \quad \text{and} \quad \frac{\partial}{\partial\psi} = \frac{1}{\eta}\frac{\partial}{\partial\xi}. \tag{59}$$

It is then possible to satisfy the equations by expressing u, v, p, and T as power series in η with coefficients which are functions of ξ only. All the series save v are in even powers of ξ; v is in odd powers. The details of the algebra, which is tedious, are omitted. Suffice it to write down the equations satisfied by u_0, p_0, ρ_0, T_0 the terms in the various series independent of η, and by v_1 the coefficient of η in the series for v; they are

$$\rho_0 = 1, \tag{60}$$

$$p_0 = T_0, \tag{61}$$

$$u_0'' + \tfrac{1}{2}\xi u_0' = 0, \tag{62}$$

$$T_0'' + \frac{\sigma}{2\gamma}\xi T_0' = -\sigma(\gamma-1)M^2 u_0'^2, \tag{63}$$

$$\tfrac{8}{3}v_1'' + \xi v_1' - v_1 = \frac{2}{\gamma}p_0', \tag{64}$$

where dashes denote differentiations with respect to ξ.

Had we not assumed μ constant we should have obtained (60) and (61), but (63) and (64) would have been replaced by simultaneous equations for u_0 and p_0 rather than, as here, equations admitting of separate solutions.

It is not difficult to write down the equations for the later coefficients, but their solution has not been attempted and so they are not reproduced here. There seems no reason to doubt that a solution on these lines could be developed, but in the light of the subsequent discussion it scarcely seems called for.

The solutions of our equations are

$$u_0 = 1 - \operatorname{erf}\tfrac{1}{2}\xi, \tag{65}$$

$$\begin{aligned} p_0 = T_0 &= 1 + \sigma\frac{(\gamma-1)}{\pi}M^2 \int_\xi^\infty e^{-\sigma\zeta^2/4\gamma} \int_0^\zeta e^{(\sigma-2\gamma)\eta^2/4\gamma}\, d\eta d\zeta \\ &= 1 + \sigma(\gamma-1)M^2\left[\frac{\gamma}{\pi(2\gamma-\sigma)}\right]^{\frac{1}{2}} \int_\xi^\infty e^{-\sigma\zeta^2/4\gamma} \operatorname{erf}\left[\left(\frac{2\gamma-\sigma}{4\gamma}\right)^{\frac{1}{2}}\zeta\right] d\zeta, \end{aligned} \tag{66}$$

$$v_1 = \frac{6\sigma(\gamma-1)M^2}{\pi(3\gamma-4\sigma)}\,\xi \int_\xi^\infty \left\{\frac{e^{-\sigma\zeta^2/4\gamma}}{\zeta^2} \int_0^\zeta e^{(\sigma-2\gamma)\eta^2/4\gamma}\, d\eta - \frac{e^{-3\zeta^2/16}}{\zeta^2} \int_0^\zeta e^{-5\eta^2/16}\, d\eta\right\} d\zeta. \tag{67}$$

We may notice that, at the wall the integral for p_0 can be evaluated to give

$$(p_0)_W = 1+\frac{2}{\pi}\gamma(\gamma-1)M^2\left(\frac{\sigma}{2\gamma-\sigma}\right)^{\frac{1}{2}}\tan^{-1}\left(\frac{2\gamma-\sigma}{\sigma}\right)^{\frac{1}{2}}, \tag{68}$$

which, as already anticipated, agrees with (55) when $\sigma = \frac{3}{4}$ and provides a useful check.

With $\sigma = 0{\cdot}7$ and $\gamma = 1{\cdot}4$, the values for air, we have

$$(p_0)_W = 1+0{\cdot}2155M^2.$$

Table 2 gives the variations of p_0 and v_1 with ξ.

TABLE 2

ξ	$(p_0-1)/M^2$	v_1/M^2
0·00	0·2155	0·0000
0·98	0·1775	0·0214
1·96	0·1032	0·0355
2·94	0·0458	0·0218
3·92	0·0163	0·0088
4·90	0·0049	0·0025
5·88	0·0012	0·0004

It is, of course, not possible to determine precisely the range of validity of the solution obtained without determination of higher terms in the series. All we can say is that it will be valid for sufficiently small times t and that these times will be measured in relation to ν_s/a_s^2 or ν_s/U^2. It may, in fact, be shown that it is always the smaller of these two quantities which is the relevant comparison time.

5. Discussion

Let us consider, in the first place, how the results we have obtained fit into the pattern of any complete solution of our equations. It is very likely that, as Carrier and Lin (**4**) found in their considerations of the incompressible flow past a semi-infinite plane, the flow can be split up, effectively, into four regions, overlapping in parts, as follows:

(i) a solution useful at small times;

(ii) a boundary-layer solution valid for large times and $y = O(t^{\frac{1}{2}})$;

(iii) a region of effectively inviscid flow;

(iv) an intermediate region in which conditions are complicated and in which numerical methods of solution seem to be the only possibility.

(i) is the solution given in section 4 above; (ii) has been given by Illingworth (**2**). (iii) and (iv) have not been attempted here, though considerations similar to those of our section 3 could possibly be used to extend (iii) and join it on to the other flow regions.

It is interesting to notice that, in the initial solution, the density remains

unaltered and the pressure variations, in consequence, are identical with the temperature variations. In contrast, in the boundary-layer solution the pressure is effectively the undisturbed pressure, whilst the density is then the inverse of the temperature. The velocity normal to the plane in the initial solution is proportional to $t^{\frac{1}{2}}$ and rises from zero at the plane to a maximum and then falls off to zero. The density variations determined in the boundary-layer solution imply a velocity normal to the plane proportional to $t^{-\frac{1}{2}}$; it rises from zero at the plane to a maximum at the edge of the layer. The variation of this velocity outside the layer and in particular the way it falls off to zero are not determined, of course, by boundary-layer considerations and require a more detailed investigation of the reaction of the viscous layer on the flow outside.

In the linearized solution of section 3, which is valid for small Mach numbers, the result in (50) is the formal expression for the pressure over all the regions (i)–(iv) though it is not in a form useful for computation. We have seen that further study of integrals of the type (51) is required before this aspect can be considered complete. Nevertheless the most interesting physical feature is the pressure at the wall, and that has been obtained in readily calculable form in (54) and is tabulated in Table 1. The general considerations of L.C.T. (**1**) regarding the form of the solution of equations of the type (33) when added to our specific results give a reasonable appreciation of the whole field. Dissipation is important only in the region $y = O(t^{\frac{1}{2}})$, where, in the light of the pressure calculated at the wall, we should expect a rapid rise of pressure initially followed by a steady decrease to the undisturbed value. These pressure variations will be communicated to the flow outside this layer along the sub-characteristics of (33) with, in addition, a dispersive effect which will arise through viscous action. The velocity of outflow v, once p is determined everywhere, is determined by (20), which is effectively a diffusion equation with a known distribution of sources. Outside the layer in which dissipation is important, these changes, too, will be propagated along the sub-characteristics of (33) with the dispersive effect of viscosity superposed.

So much for the actual solution of our equations. Now let us turn to the numerical results of section 3; their most interesting and striking feature, apart from the instantaneous rise, is the rapidity with which the pressure falls off at the plane. For air under standard conditions $\nu_s = 0{\cdot}132$ cm.^{2}/sec., $a_s = 3{\cdot}3 \times 10^4$ cm./sec., so that ν_s/a_s^2 is of the order of 10^{-10} sec. Referring to Table 1 and (56), we see that the pressure has fallen to one-half its initial value in about 5×10^{-10} sec. and to one-tenth in about 10^{-8} sec.

Now the time between successive molecular collisions is of the order of 10^{-10} sec. Over such times the treatment of the gas as a continuum (the

assumption on which the classical equations we have used is based) is no longer possible and we must expect the first half of our pressure changes to require re-examination in the light of molecular considerations. Furthermore, since our equations are based on energy considerations relaxation effects must also be taken into consideration (see Gunn **5**). The relaxation time for air, which is of the order of 10^{-4} sec., represents the order of the time for the 'inert' degrees of freedom to respond to pressure changes; the 'active' degrees of freedom respond much more rapidly, in times of the order of 10^{-10} sec. We must, therefore, expect modification of our results up to times of the order of 10^{-4} sec. The early effects in the first half of our predicted pressure-time result are probably the most marked, though the others are likely to be by no means negligible.

For times greater than 10^{-4} sec., our equations would be expected to be valid and we may notice that the excess pressures they predict are then negligible. Without therefore going into molecular considerations in detail it is reasonable to infer that the onset of the motion is marked by rapid pressure changes which die out in the order of 10^{-4} sec. Though in itself this inference is not conclusive evidence, it does point strongly to the fact that the boundary-layer solution is valid down to times of this order.

The importance of molecular considerations, moreover, makes it appear doubtful whether any great effort should be devoted to filling in the gaps in the solution of the classical equations for this particular problem.

It is very tempting to draw conclusions about the steady flow past a semi-infinite plane in the same way that Rayleigh did for the associated incompressible problem by replacing t in our considerations by x/U, where x is distance from the leading edge. We might, in the very rapid pressure changes, claim to see indications of the shock wave which it seems likely must exist there. However, although there may be many points of similarity, this approach which, as Squire has pointed out, corresponds to linearizing the inertia terms in the steady flow problem cannot, I feel, immediately admit of this extension. In the unsteady flow dissipation is the sole cause of outflow normal to the plate and of the pressure variations. In the steady flow, changes of velocity parallel to the plane also contribute to this outflow and in their turn also induce pressure changes. In fact an examination of the linearized equation for the steady flow shows evidence of some dispersion of pressure effects on this account though they may not (and indeed from experimental results it appears unlikely that they will) be sufficient to prevent a still very rapid change in pressure. Further examination is reserved to a later paper.

REFERENCES

1. P. A. Lagerstrom, J. D. Cole, and L. Trilling, *Problems in the Theory of Compressible Fluids*, G.A.L.C.I.T. Report (1949).
2. C. R. Illingworth, *Proc. Camb. Phil. Soc.* **46** (1950), 603.
3. H. Jeffreys and B. Jeffreys, *Methods of Mathematical Physics* (Cambridge, 1950), chaps. 7, 12, 14.
4. G. F. Carrier and C. C. Lin, *Quart. of Applied Mathematics*, **6** (1948), 63.
5. J. C. Gunn, *Relaxation Time Effects in Gas Dynamics*. Rep. Memor. Aero. Res. Council, No. 2338 (1947).

ON A PROBLEM OF INTERACTION OF PLANE WAVES OF FINITE AMPLITUDE INVOLVING RETARDATION OF SHOCK-FORMATION BY AN EXPANSION WAVE

By P. M. STOCKER

(*Department of Mathematics, The University, Manchester*)

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SUMMARY

The interaction, in an inviscid perfect gas with constant specific heats in the ratio 5/3, of a receding simple compression wave and an advancing, centred, simple expansion wave is calculated by the method of characteristic coordinates, the interaction being taken to begin before a limit line (corresponding physically to the formation of a shock wave) is formed in the compression wave. The time is determined by which the interaction delays the formation of a limit line. It is found that shock formation can be delayed considerably, but not indefinitely. (For a formula for the time at which shock formation occurs in a simple wave, see the Appendix.)

1. Introduction

THE one-dimensional, isentropic, unsteady flow of an inviscid fluid in a region which is not a simple wave has been studied by several authors, in particular by Riemann (**1**), and by Love and Pidduck (**2**). In many problems of this kind shock formation could not occur; there are, however, others of considerable interest, in which the possibility of shock formation may be studied.

The problem considered here is that of the interaction of two simple waves, one compressive and one expansive. The calculations are made for the case in which the compression wave is receding, with its front passing through $x = 0$ at time $t = 0$ and the expansion wave is advancing, being centred at $x = -2c$ at $t = 0$ (Fig. 1). Between the waves the gas is assumed to be initially uniform and at rest.

The straight characteristics in the compression wave are converging and form an envelope (a limit line); the physical interpretation is that a shock wave must appear when or before the limit line is formed. The presence of the rarefaction wave delays the formation of this limit line.

It is convenient to think of the simple waves as caused by piston motions. But this implies that reflection will eventually occur at the piston causing the compression wave and possibly also at the other piston. Such reflections are not relevant to the problem to be considered, and we do not wish to take account of them. By assuming that the velocity of the

left-hand piston is not less than the escape speed of the gas, we may ensure that no reflection occurs at this piston. The compression wave, on the other hand, may be regarded as generated by a piston starting to move at a time earlier than $t = 0$, from a position with $x > 0$, and by choosing this value of x sufficiently large we may postpone reflection from this piston for any length of time.

As a convenient method, however, of defining the compression wave, we shall specify the equation of the particle path through $x = 0$, $t = 0$. This curve may be regarded as the path of a piston starting to move at $x = 0$ at time $t = 0$ with zero initial velocity, and if, for instance, the simple wave contains the particle path $x = -\frac{1}{2}\nu t^2$, where ν is a constant, we may refer to it as 'of the type caused by a piston moving with uniform acceleration $-\nu$'.

In section 2 characteristic coordinates are introduced, the dependent variable to be used is chosen, and the equation to be solved is found for a perfect gas with constant specific heats in the ratio γ, together with its solution in general terms when $n = \frac{1}{2}(\gamma+1)/(\gamma-1)$ is a positive integer. In section 3 the solution is expressed in terms of the boundary values, for any piston motion defining the compression wave, on the assumption that $\gamma = 5/3$, i.e. $n = 2$. In section 4 this solution is used to write down the equations of the singular lines in the characteristic plane, which correspond to limit lines in the (x, t)-plane; these equations are then specialized for (i) the case when the piston defining the compression wave has uniform acceleration $-\nu$, and (ii) for the case when it moves under the law $x = -\eta t^3$, where η is a positive constant. In section 5 the position at which a limit line is first formed is considered in the case of a uniformly accelerated piston. In order that it should not occur before interaction starts, the parameter $\lambda = 36(\gamma-1)^2 L\nu/a_0^2$ (where a_0 is the velocity of sound in the undisturbed gas) must be less than 12; the singular lines are computed (and shown in Fig. 2) for $\lambda = 10$. The position x_s and time t_s at which a limit line first appears (corresponding physically to the appearance of a shock wave) are determined in section 5.1 for all integral values of the parameter λ from 0 to 12 inclusive; the ratios x_s/x_c and t_s/t_c of these coordinates and times to the corresponding values for the compression wave alone, in the absence of interaction, are shown in Table 1 (section 5.1 below) to lie between 9·7 and unity, and 3·73 and unity respectively. The case of a piston moving under the law $x = -\eta t^3$ is treated in section 5.2; the position at which the shock first appears is computed for the values 200 and 100 of the parameter $N = 360L(3\eta)^{\frac{1}{2}}s_0^{-\frac{3}{2}}$.

A simple formula for the time at which a shock forms in a receding simple wave is established in the Appendix.

2. The equations of motion

The equations governing the one-dimensional, isentropic, unsteady flow of inviscid gas, in terms of Eulerian coordinates x and t, are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0, \tag{1}$$

and

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0, \tag{2}$$

where $u(x,t)$, $\rho(x,t)$, and $p(x,t)$ denote respectively the velocity, density, and pressure of the fluid at the point x at time t. Introduce the Riemannian characteristic variables r and s, defined by

$$r=\frac{1}{2}\int\limits_0^\rho \frac{a}{\rho}\,d\rho+\tfrac{1}{2}u, \qquad s=\frac{1}{2}\int\limits_0^\rho \frac{a}{\rho}\,d\rho-\tfrac{1}{2}u, \tag{3}$$

where a is the local velocity of sound and is equal to $(\partial p/\partial\rho)^{\frac{1}{2}}$, the derivative being taken with constant entropy. The equations become

$$\frac{\partial r}{\partial t}+(u+a)\frac{\partial r}{\partial x}=0, \tag{4}$$

and

$$\frac{\partial s}{\partial t}+(u-a)\frac{\partial s}{\partial x}=0. \tag{5}$$

Thus

$r=$ constant on any 'positive' characteristic $dx=(u+a)\,dt$,

and

$s=$ constant on any 'negative' characteristic $dx=(u-a)\,dt$.

Equations (4) and (5) may be inverted, and yield the equations

$$\frac{\partial x}{\partial r}=(u-a)\frac{\partial t}{\partial r}, \tag{6}$$

$$\frac{\partial x}{\partial s}=(u+a)\frac{\partial t}{\partial s}. \tag{7}$$

If, however, the Jacobian

$$J=\frac{\partial(x,t)}{\partial(r,s)}=-2a\frac{\partial t}{\partial r}\frac{\partial t}{\partial s} \tag{8}$$

is zero or infinite at one or more points in the region considered, the correspondence of the regions of the (x,t)-plane and the (r,s)-plane will not be regular and $(1,1)$.

Eliminating x we obtain the equation for t:

$$2a\frac{\partial^2 t}{\partial r\partial s}+\frac{\partial(u+a)}{\partial r}\frac{\partial t}{\partial s}-\frac{\partial(u-a)}{\partial s}\frac{\partial t}{\partial r}=0;$$

u and a are known functions of r and s and this is a linear equation for t.

For a perfect gas with constant specific heats in the ratio γ, $p = A\rho^{\gamma}$, where A is a constant for isentropic flow,

$$r = a/(\gamma-1)+\tfrac{1}{2}u, \qquad s = a/(\gamma-1)-\tfrac{1}{2}u, \tag{9}$$

and the equation becomes

$$\frac{\partial^2 t}{\partial r \partial s}+\frac{n}{(r+s)}\left(\frac{\partial t}{\partial r}+\frac{\partial t}{\partial s}\right) = 0, \tag{10}$$

where
$$n = \tfrac{1}{2}(\gamma+1)/(\gamma-1). \tag{11}$$

This equation was studied by Riemann; his well known method for the solution of linear partial differential equations was, in fact, developed to solve it. In general, the solution may be expressed in terms of integrals involving hypergeometric functions, but when n is a positive integer the hypergeometric functions can be expressed in finite terms and the solution can be written in the form

$$t = \left[\frac{1}{(r+s)}\left(\frac{\partial}{\partial r}+\frac{\partial}{\partial s}\right)\right]^{n-1}\left[\frac{\phi_1(r)+\phi_2(s)}{(r+s)}\right],$$

where ϕ_1 and ϕ_2 are arbitrary functions to be determined from the boundary conditions.

The complexity of the solution increases with increasing n, and it is easier to use the dependent variable w, introduced by Riemann, and defined by the equations

$$\frac{\partial w}{\partial r} = x-(u+a)t, \qquad -\frac{\partial w}{\partial s} = x-(u-a)t, \tag{12}$$

in place of t, since w satisfies the equation

$$\frac{\partial^2 w}{\partial r \partial s}+\frac{n-1}{r+s}\left(\frac{\partial w}{\partial r}+\frac{\partial w}{\partial s}\right) = 0. \tag{13}$$

This equation is of the same form as equation (10), but with $n-1$ in place of n, so, for positive integral n, the solution is

$$w = \left[\frac{1}{(r+s)}\left(\frac{\partial}{\partial r}+\frac{\partial}{\partial s}\right)\right]^{n-2}\left[\frac{F(r)+f(s)}{r+s}\right], \tag{14}$$

where F and f are arbitrary functions, and these are more easily determined from the boundary conditions.

In the following work an integral value of n is assumed; since we are mainly concerned with qualitative results, the value $n = 2$, $\gamma = 5/3$, is chosen for simplicity. (A convenient method of determining $f(s)$ and $F(r)$ for larger integral n is given by Taub in (**3**).)

3. Boundary conditions

The interaction of the simple waves commences at $t = -L/a_0$, $x = -L$, and leads to the formation of a compound wave in the region $P'O'Q'$ (Fig. 1).

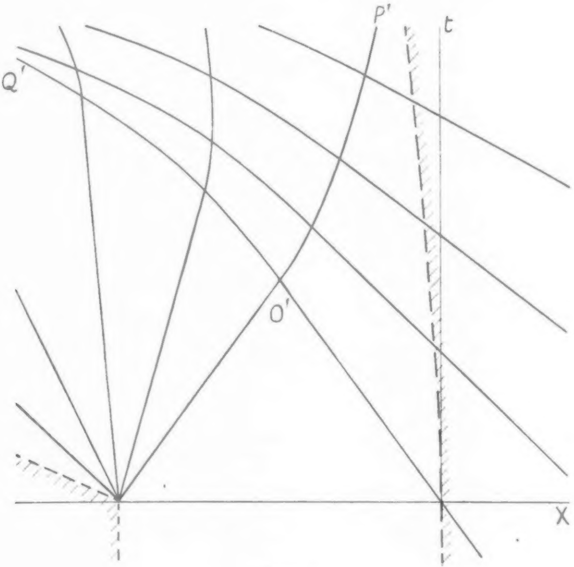

FIG. 1.

The solution in this region is of the form (14), with $n = 2$, that is

$$w = \frac{1}{(r+s)}[F(r)+f(s)], \tag{15}$$

and $f(s)$ and $F(r)$ must be determined from conditions at the wave fronts $O'P'$ and $O'Q'$, where x and t are continuous functions of r and s.

In the compression wave the negative characteristics are straight and have the equation

$$x = (u-a)t+T(s), \tag{16}$$

where $T(s)$ is a function determined by the piston motion defining the wave. In this wave r is a constant and equal to its value on $O'P'$, which is $r_0 = s_0 = a_0/(\gamma-1)$.

The equation of the positive characteristics in the expansion wave is

$$x = (u+a)t-2L; \tag{17}$$

and in this wave $s = s_0$. Thus at the advancing front, $O'P'$, we have

$$[\partial w/\partial s]_{r=s_0} = -[x-(u-a)t]_{r=s_0} = -T(s),$$

and at the receding front, $O'Q'$,

$$[\partial w/\partial r]_{s=s_0} = [x-(u+a)t]_{s=s_0} = -2L.$$

Choosing $F(r_0) = f(s_0) = 0$ and integrating, we have

$$f(s) = -(s+s_0)\int\limits_{s_0}^{s} T(s)\,ds, \tag{18}$$

$$F(r) = 2L(s_0^2-r^2), \tag{19}$$

and thus
$$w = \frac{2L(s_0^2-r)-(s+s_0)\int\limits_{s_0}^{s} T(s)\,ds}{(r+s)}. \tag{20}$$

4. Singular lines

The correspondence between the region $P'O'Q'$ and the region

$$-s_0 \leqslant r \leqslant s_0, \quad s \geqslant s_0$$

in the (r,s)-plane, will not be regular and $(1,1)$ if J vanishes.† When J is zero, either $\partial t/\partial s$ or $\partial t/\partial r$ is zero, and a line in the (x,t)-plane on which either vanishes is an envelope of one family of characteristics and a cusp line of the other family (**5**). Near such a line the solution will therefore yield two different values of the velocity, at the same point and at the same time, and thus cannot represent the motion of a gas. The envelope of characteristics in the (x,t)-plane is called a limit line and the corresponding curve in the (r,s)-plane is called a singular line.

By (9) and (12), we have, for $\gamma = 5/3$,

$$x = \frac{(2s-r)\,\partial w/\partial r+(s-2r)\,\partial w/\partial s}{(r+s)}, \qquad t = -\frac{3}{2}\frac{1}{(r+s)}\left(\frac{\partial w}{\partial r}+\frac{\partial w}{\partial s}\right), \tag{21}$$

and by use of (15) we find that the equation of the singular line $\partial t/\partial s = 0$ is

$$(r+s)^2 f''(s)-2\big(F'(r)+2f'(s)\big)(r+s)+6\big(F(r)+f(s)\big) = 0,$$

where primes denote differentiation with respect to the argument. The equation of the other singular line $\partial t/\partial r = 0$ is, similarly,

$$(r+s)^2 F''(r)-2\big(2F'(r)+f'(s)\big)(r+s)+6\big(F(r)+f(s)\big) = 0.$$

† The correspondence will also not be regular if J is infinite. If J is infinite either $\partial r/\partial x$ or $\partial s/\partial x$ must be zero; in this case we have a singularity in the mapping of the (x,t)-plane upon the (r,s)-plane. The solution given above fails, though it can be modified to yield the correct solution. In this work, however, we assume that, for $t > 0$, the piston motion defining the compression wave is always one that has negative acceleration; in this case zeros of $\partial r/\partial x$ and $\partial s/\partial x$ cannot occur inside $P'O'Q'$ (**5**).

Substituting for $F(r)$ and its derivatives from (19), we find that the equations become respectively

$$r^2(f''-4L)+r(2sf''-4f'+8Ls)+s^2f''-4sf'+6f+12Ls_0^2 = 0, \tag{22}$$

and
$$r = \frac{sf'-3f-6Ls_0^2+2Ls^2}{4Ls-f'}. \tag{23}$$

4.1. As stated in the introduction, it is convenient to think of the compression wave as being caused by a piston which starts to move from $x = 0$ at time $t = 0$. We now specialize the formula obtained in the previous paragraph for two particular piston motions. Suppose firstly that the equation of the piston curve is $x = -\frac{1}{2}\nu t^2$, where ν is a positive constant. Let $x_p(s)$ and $t_p(s)$ be the values of x and t at the point at which a typical straight characteristic, $s =$ constant, meets the piston curve. Then

$$-\nu t_p = u = s-s_0; \qquad t_p = \frac{s-s_0}{\nu}; \qquad x_p = -\frac{(s-s_0)^2}{2\nu}.$$

Substitute in (16) to obtain

$$T(s) = (5s^2-6ss_0+s_0^2)/6\nu; \tag{24}$$

and with this value of $T(s)$, from (18),

$$f(s) = -(5s^4-4s^3s_0-6ss_0^2+4ss_0^3+s_0^4)/18\nu. \tag{25}$$

Substitute for $f(s)$ in (22) and let

$$\lambda = 36L\nu/s_0^2. \tag{26}$$

The equation (22) of the singular line, $\partial t/\partial s = 0$, corresponding to an envelope of the negative characteristics, is then found to be

$$r^2(60s^2-24ss_0-12s_0^2+2\lambda s_0^2)+r(40s^3+24ss_0-16s_0^3-4\lambda ss_0^2)+$$
$$+10s^4+8ss_0^3+6s_0^4-6\lambda s_0^4 = 0. \tag{27}$$

By substituting for $f(s)$ in (23), we find that the equation of the other singular line is

$$r = \frac{5s^4+6s^2s_0^2-\lambda ss_0^3-8ss_0^3+3\lambda s_0^4-3s_0^4}{20s^3+2\lambda ss_0^2-12s^2s_0-12ss_0^2+4s_0^3}. \tag{28}$$

Secondly, we obtain the solution in the case when the piston law is $x = -\eta t^3$, where η is a positive constant. By the same method as used above, we find that

$$T(s) = \frac{(s-s_0)^{\frac{1}{2}}(3s-s_0)}{(27\eta)^{\frac{1}{2}}}, \tag{29}$$

and
$$f(s) = -4K(9s^2+10ss_0+s_0^2)(s-s_0)^{\frac{3}{2}}, \tag{30}$$

where
$$K = \frac{1}{90(3\eta)^{\frac{1}{2}}}. \tag{31}$$

5. The position of shock formation

The physical interpretation of the appearance of a limit line is that the isentropic flow breaks down and a shock is formed. The best estimate, in the present stage of knowledge, for the position and time at which the shock wave is formed is the position and time at which a limit line first appears. We therefore calculate the lowest value of t on the limit line.

The singular lines given by (27) and (28), for $\lambda = 10$ are shown in Fig. 2.

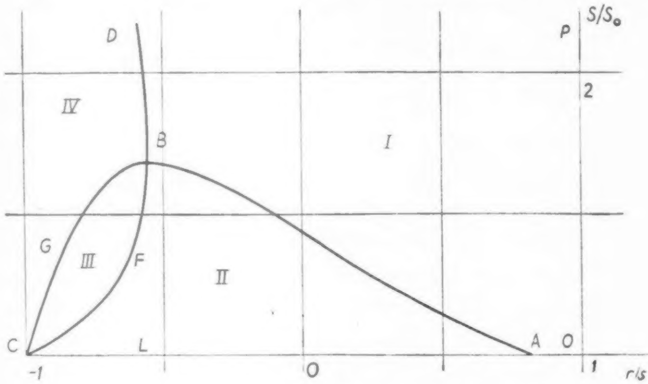

FIG. 2.

The singular line ABC is the line $\partial t/\partial s = 0$, and CFD is the line $\partial t/\partial r = 0$. In region I, $\partial t/\partial s$ and $\partial t/\partial r$ have the same sign as in the simple waves; that is, $\partial t/\partial r < 0$, $\partial t/\partial s > 0$. Moreover, $\partial^2 t/\partial r \partial s \neq 0$ (except at B); hence in

$$\text{Region II:}\quad \partial t/\partial s < 0,\quad \partial t/\partial r < 0,$$

$$\text{Region III:}\quad \partial t/\partial s < 0,\quad \partial t/\partial r > 0,$$

$$\text{Region IV:}\quad \partial t/\partial s > 0,\quad \partial t/\partial r > 0.$$

Therefore, the time has its minimum value on CFD, and its maximum value on ABC, at B. On the singular line BGC, which corresponds to an envelope of negative characteristics in the (x,t)-plane, we have $\partial t/\partial s = 0$, $\partial t/\partial r > 0$, and we find that $t \to -\infty$ as we approach C along this line. Hence the earliest appearance of a limit line is at $t = -\infty$, but clearly this can have no physical significance, since interaction did not begin until $t = L/a_0$. We must restrict ourselves to that part of the (x,t)-plane in which our isentropic solution can have physical significance, that is the region which can be reached from the simple waves without crossing a singularity. This region corresponds to the region in the (r,s)-plane bounded by the line $r = s_0$, a segment of the line $s = s_0$, and the singular lines AB and BD. Shock formation will occur at the lowest value of t on

the limit lines corresponding to AB and BD, i.e. at the point corresponding to A. The isentropic solution which we have found in section 4.1 will hold only in the region of the (r, s)-plane bounded by OA, $r = s_0$, and the curve through A which corresponds to the shock path.

5.1. We now investigate the position of shock formation for compression waves of the type caused by a uniformly accelerated piston, for general λ. It is shown in the Appendix that, in the absence of interaction, a shock would form at the head of the compression wave at $x = -\frac{1}{3}s_0^2/\nu$; in order that the shock should not form until after the interaction has commenced we must choose

$$L < \tfrac{1}{3}s_0^2/\nu, \quad \text{that is, } \lambda < 12.$$

To determine the position of shock formation we have to find the lowest value of t on the singular line $\partial t/\partial s = 0$, as in the case $\lambda = 10$.

If we take s as a parameter along the singular line, and let the equation of the singular line be $r = R(s)$, we have, differentiating t along the singular line,

$$\frac{dt}{ds} = \left(\frac{\partial t}{\partial s}\right)_r + \left(\frac{\partial t}{\partial r}\right)_s \frac{dR}{ds} = \left(\frac{\partial t}{\partial r}\right)_s \frac{dR}{ds}.$$

The zero of $\partial t/\partial r$ can be shown to correspond to a maximum of t on the singular line, as in the case $\lambda = 10$, and to occur at an end-point of the singular line segment under consideration; let this point be $B(\lambda)$. A minimum of t on the line can occur only at a point where $dR/ds = 0$; if there is no such point the lowest value of t will occur at $s = s_0$.

We have found numerically, for $\lambda = 10$, that $dR/ds < 0$ on the segment AB of the singular line. Suppose a zero of dR/ds occurs for $\lambda = \lambda^*$ inside the corresponding segment, but for no value of λ between λ^* and 10. Then, dR/ds being a continuous function of λ (except at $B(\lambda)$), $dR/ds \leqslant 0$ on the segment; moreover, dR/ds is a differentiable function of s (except at $B(\lambda)$), and hence d^2R/ds^2 also vanishes where $dR/ds = 0$.

From (27) we find that $dR/ds = d^2R/ds^2 = 0$ implies $R = -s$; but $r = -s$ is the equation of the vacuum line, and hence there is no point on the segment of the singular line at which $d^2R/ds^2 = dR/ds = 0$. Thus as we increase or decrease from the value $\lambda = 10$ a zero of dR/ds must first occur at $s = s_0$.

Differentiation of (27) with respect to s yields

$$R^2(120s-24s_0)+R(120s^2+24s_0^2-4\lambda s_0^2)+40s^3+8s_0^3+$$
$$+\frac{dR}{ds}\{2R(60s^2-24ss_0-12s_0^2+2\lambda s_0^2)+40s^3+24ss_0^2-4\lambda ss_0^2-16s_0^3)\} = 0,$$

whence,
$$\left(\frac{dR}{ds}\right)_{s=s_0} = -\frac{24R^2(s_0)+s_0\,R(s_0)(36-\lambda)+12s_0^2}{s_0\{R(s_0)(12+\lambda)+12-\lambda\}}. \tag{32}$$

Also, from (22), $$R(s_0) = \frac{3\lambda-12}{12+\lambda}s_0 \quad \text{or} \quad -s_0. \tag{33}$$

Substituting the first of these two values in (32), we find that

$$(dR/ds)_{s=s_0} = 0$$

when $$\left(\frac{\lambda}{12}\right)^2 - \frac{26}{3}\left(\frac{\lambda}{12}\right)+1 = 0.$$

There is one root of this equation in the range $(0, 12)$, namely 1·404, and for this value of λ, $dR/ds = 0$ on the limit line at $s = s_0$, i.e. $\lambda^* = 1{\cdot}404$.

Thus, if λ lies in the range $12 \geqslant \lambda > 1{\cdot}404$ shock formation occurs at the head of the compound wave; if $1{\cdot}404 > \lambda \geqslant 0$ shock formation occurs inside the wave.

The values, r_s and s_s, for which t has its lowest value on the singular line segment considered are given in Table 1 for integral values of λ. For $1{\cdot}404 < \lambda \leqslant 12$, $s = s_s$ and r_s is given by (33); for $\lambda = 0$ and $\lambda = 1$, the singular line has been computed to determine the zero of dR/ds. The corresponding values, x_s and t_s, of x and t, which provide an estimate for the position and time at which a shock is formed, are also given in Table 1; x_c and t_c denote the position and time at which a shock would be formed if the rarefaction wave were absent (cf. Appendix). The values of x_s and t_s are found from (20), (21), and (25) by substituting $r = r_s$, $s = s_s$; for $1{\cdot}404 < \lambda \leqslant 12$

$$x_s = -L(3\mu^2-4\mu+6)/8, \qquad t_s = 3L(1-\mu^2)/8s_0,$$

where $\mu = 12/\lambda$.

TABLE 1

λ	r_s/s_0	s_s/s_0	t_s/t_c	x_s/x_c
0	−0·777	1·319	3·731	9·702
1	−0·668	1·137	3·123	7·465
2	−0·429	1·000	2·042	4·124
3	−0·200	1·000	1·563	2·688
4	0·000	1·000	1·333	2·000
5	0·177	1·000	1·204	1·629
6	0·333	1·000	1·125	1·375
7	0·474	1·000	1·074	1·224
8	0·600	1·000	1·042	1·125
9	0·714	1·000	1·021	1·063
10	0·818	1·000	1·008	1·025
11	0·913	1·000	1·002	1·006
12	1·000	1·000	1·000	1·000

The case $\lambda = 0$ is a limiting case; λ may tend to zero in two ways. If $L \to 0$ and ν remains finite x_s and t_s tend to finite values, thus we cannot postpone shock formation indefinitely if the piston has non-zero acceleration; if $\nu \to 0$ and L remains finite $x_s \to -\infty$ and $t \to \infty$.

The values in Table 1 suggest that, for the values of λ for which the shock forms at the head of the wave, the ratio $t_c(x_s-x_c)/x_c(t_s-t_c)$ is independent of λ, and is equal to 3. It can be shown that this result is true for any compression wave if (*a*) $\gamma = 5/3$, (*b*) the shock is formed at the head of the compound wave, and (*c*) if in the absence of interaction the shock would form at the head of the compression wave.

5.2. The position of shock formation when the compression wave is such that, in the absence of interaction, shock formation would occur inside the wave has also been studied in one special case. The case considered is that corresponding to a piston moving under the law $x = -\eta t^3$, $\eta > 0$. The parameter corresponding to λ is $N = 360L(3\eta)^{\frac{1}{2}}s_0^{-\frac{3}{2}}$, and to ensure that shock formation does not occur before interaction it is sufficient to take $N < 217$. In this case, in the absence of interaction shock formation will occur on $s = 11s_0/9$.

The equation of the singular line in the (r, s)-plane is found by substituting the expression for $f(s)$ from equation (30) in (22). This line has been computed for the values $N = 200$ and $N = 100$ and the zero of dR/ds determined. The results are given in Table 2.

TABLE 2

N	s_s/s_0	r_s/s_0	t_s/t_c	x_s/x_c
100	1·261	0·315	1·100	1·274
200	1·215	0·850	1·018	1·043

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APPENDIX

Shock Formation in a Simple Wave

It is shown in (**4**) that when a piston moves into a gas at rest there is a limit point on the front of the wave generated by the piston if and only if the initial acceleration of the piston is non-zero. This need not, however, be the first limit point to appear during the motion, and hence need not provide an estimate for the time and position at which a shock wave first appears. A simple formula for the time at which a limit point first appears may be established as follows.

Let the speed of sound in the gas at rest be a_0 and let a receding wave move into it. Then $r = s_0 = a_0/(\gamma-1)$ by (9), and the equation of the straight negative characteristic is

$$x = (u-a)t+T(s). \tag{34}$$

The limit line is an envelope of the characteristics, and hence it is given by (34) together with $dx/ds = dt/ds = 0$, i.e. by (34) and

$$t = 2T'(s)/(\gamma+1). \tag{35}$$

If the motion of the piston (or indeed, any particle path) is given by $x = x_p(s)$ and $t = t_p(s)$, then

$$T(s) = x_p(s)-(u-a)t_p(s),$$

and $dx_p/dt_p = u$, the speed of the piston. Moreover, by (9) since $r = s_0$,

$$dt_p/ds = -dt_p/du = -1/b_p,$$

where b_p is the acceleration of the piston. Hence

$$T'(s) = \tfrac{1}{2}(\gamma+1)t_p-2/b_p,$$

and, by (35), the time at which any one of the straight characteristics s = constant meets the limit line is

$$t_l = t_p-2a/\{(\gamma+1)b_p\}.$$

The earliest time, t_c, at which a limit point appears is therefore equal to the *lowest value of*

$$t-2a/\{(\gamma+1)b\}$$

on the piston curve (where t is the time, a the speed of sound, and b the acceleration of the piston), and this limit point lies on the straight characteristic through the piston curve where the lowest value occurs. If $t-2a/\{(\gamma+1)b\}$ is a decreasing function of t on the piston curve at the point where the piston starts its motion, the shock will not form at the front of the wave, even if the initial acceleration is non-zero.

Similarly, the time at which a limit point first appears in an advancing, simple compression wave is equal to the lowest value of $t+2a/\{(\gamma+1)b\}$ on the piston curve.

When $x_p = \frac{1}{2}\nu t_p^2$ and $\gamma = 5/3$, $T(s)$ is given by (24), and by (35),

$$t_c = \tfrac{1}{2}s_0/\nu.$$

From (34), the corresponding value of x is found to be

$$x_c = -\tfrac{1}{3}s_0^2/\nu.$$

When $x_p = -\eta t_p^3$ and $\gamma = 5/3$, $T(s)$ is given by (29), and by (35),

$$t_c = \tfrac{1}{8}(3\eta)^{-\frac{1}{2}}(9s-7s_0)(s-s_0)^{-\frac{1}{2}}.$$

This has a minimum when $s = 11s_0/9$, and

$$t_c = \tfrac{3}{4}(a_0/\eta)^{\frac{1}{2}},$$

$$x_c = -23(a_0^3/\eta)^{\frac{1}{2}}/36.$$

REFERENCES

1. B. RIEMANN, 'Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite', *Ges. Werke* (Leipzig, 1876), 145–64.
2. A. E. H. LOVE and F. B. PIDDUCK, 'Lagrange's Ballistic Problem', *Phil. Trans. Roy. Soc.* **222** (1922), 167–226.
3. H. TAUB, 'The Interaction of Progressive Rarefaction Waves', *Annals of Math.* (2), **47** (1946), 811–28.
4. R. COURANT and K. O. FRIEDRICHS, *Supersonic Flow and Shock Waves* (New York, 1948), 110–15.
5. P. M. STOCKER and R. E. MEYER. (Unpublished.)

ON THE IMPULSIVE MOTION OF A FLAT PLATE IN A VISCOUS FLUID

By K. STEWARTSON

(*Department of Mathematics, The University, Bristol*)

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SUMMARY

The motion in the boundary layer which arises when a semi-infinite flat plate is impulsively started from rest with velocity U is investigated. It is found that the velocity field is independent of x, the distance along the plate if $Ut < x$, and at $Ut = x$ the flow has an essential singularity, thereafter depending on x as well as t. As $t \to \infty$ the influence of t is shown to die out exponentially. Solutions are also obtained using two approximate methods and they both agree in requiring the velocity to be independent of x for a finite time.

The same problem is also solved for a compressible fluid. It is found that in this case the outflow is increased and this may lead to the formation of two compression waves.

1. Introduction

THIS paper is concerned with the motion of the fluid in the boundary layer associated with a semi-infinite plate, which starts to move impulsively along its length with constant velocity U in a fluid at rest at infinity. Previous papers on this subject, of which Goldstein has given a connected account (**1**), consider the impulsive motion of cylinders for which the relative velocity U_1 of the fluid outside the boundary layer is not constant, and in which separation occurs fairly quickly, before the boundary layer has had time to settle down to its steady state. The solution is obtained in the following way. Initially the boundary layer has zero thickness and therefore at the beginning of the motion the diffusion is far greater than the convection and the influence of the pressure gradient. Consequently the velocity in the boundary layer is a function of y, the normal distance from the surface of the cylinder, and the time t, multiplied by the relative main stream velocity, calculated from the inviscid theory. A second approximation is then obtained by substituting it back into the boundary-layer equations. The process may be repeated as many times as desired, the only restriction being that the boundary-layer equations are invalid beyond a point of separation. A peculiar feature of the method is that if, as in the problem to be discussed in this paper, U is constant, all the terms after the first are zero, so that the leading edge of the plate apparently exerts no influence on the motion in the boundary layer. Such a conclusion is paradoxical since the flow eventually steadies down to the Blasius

solution. It is of interest, therefore, to examine more closely the solution in this case.

Before examining the full boundary-layer equations, some light may be thrown on the general nature of the flow in the boundary layer by two simple approximate methods. The first, known as Rayleigh's method, linearizes the boundary-layer equations and is only valid at the outer edge where the fluid viscosity is approximately the same as the main stream velocity. The solution it gives is independent of x, the distance from the leading edge along the plate, if $Ut < x$, and is independent of t if $Ut > x$. It suggests that the fluid velocity at (x, y) is unaffected by the leading edge of the plate if $Ut < x$, and is independent of the time of the initial pulse if $Ut > x$.

The other method, which makes use of the momentum integral, gives a solution which is independent of x if $Ut < 2{\cdot}65x$, and is independent of t if $Ut > 2{\cdot}65x$. Both these methods inevitably over-simplify the problem, but in view of the success of the momentum integral in determining the skin friction in accelerated flows, and of the validity of the Rayleigh method at the outer edge of the boundary layer, it seems likely that the change over from t to x as a principal independent variable will begin at the outer edge of the boundary layer when $Ut = x$ and will be nearly complete when $Ut = 2{\cdot}65x$, having spread down to the plate.

The boundary-layer equations for this problem are complicated and, unfortunately, the conclusions derived from an examination of them are to a certain extent tentative. Since the straightforward method of approximation breaks down, it might be expected that the flow would be independent of x for a finite time and that then x would enter by way of an essential singularity. We find that a solution of this form can be found, and that in it the critical value of t is x/u. The velocity in this solution above any point of the plate is therefore independent of x until that point passes through the original position of the leading edge. The influence of x is most marked near the outer edge of the boundary layer when $(Ut-x)/x$ is small and increases inwards with $(Ut-x)/x$. When Ut/x is large the solution as a power series fails in the same way as when Ut/x is small. In this case a solution is found with an essential singularity at $x/Ut = 0$, so that the flow would eventually steady down to the Blasius solution. It may well be that there are many other possible solutions, for none of the arguments used absolutely preclude them, but it has not been possible to find any except those with essential singularities at $Ut = x$ and $x/Ut = 0$ only. There are actually an infinite number of possible solutions with essential singularities at $x/Ut = 0$, but with one exception they all contain an oscillatory component.

The solution we find does agree with Rayleigh's method if $Ut < x$, as indeed we should expect, for the latter is then an exact solution. It is only when $Ut > x$ that the flow becomes mathematically complicated, but still the general features are as outlined in the last paragraph.

The discussion above has tacitly assumed that the fluid was incompressible, but a solution of the same form may also be obtained when compressibility effects are taken into account. If we transform y by means of

$$Y = \int_0^y \frac{\rho\, dy}{\rho_0}, \tag{1.1}$$

where ρ, ρ_0 are the densities at x, y, t, and infinity respectively, and assume that the viscosity is proportional to the absolute temperature, the stream function can be shown to satisfy the same equation whether the fluid is compressible or not. The solutions are slightly different, however, since the compressible solution is completed by transforming back from Y to y. In addition to the outflow when $Ut > x$, which occurs in the incompressible solution, there is an outflow set up at $t = 0$.

2. The statement of the problem

The problem may be regarded as one in which a velocity is impulsively set up in the main stream, the plate remaining at rest, and we shall treat it in this way here. Let us take as origin a point on the leading edge of the plate, the x-axis along its length, the y-axis perpendicular to it, and let the motion begin at $t = 0$. If u and v are the components of fluid velocity along and perpendicular to the plate, then for an incompressible fluid the boundary-layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

and

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p_1}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}. \tag{2.2}$$

When the fluid is compressible the equations are slightly different, and they are given in the last section (equations 7.3–7.6).

The velocity in the main stream is constant and equal to U relative to the plate, so that $\partial p_1/\partial x = 0$ and the boundary conditions are

$$\begin{aligned} &u = v = 0, && \text{when } y = 0,\ x > 0,\ t > 0;\\ &u = U, && \text{when } y > 0 \text{ and either } t \geqslant 0,\ x = 0, \text{ or } x \geqslant 0,\ t = 0;\\ &u \to U, && \text{as } y \to \infty,\ x \geqslant 0,\ t \geqslant 0. \end{aligned}$$

The two approximate solutions of this problem will now be given before considering the boundary-layer equations in detail.

3. Rayleigh's method

The basic assumption of this method is that the variation of velocity from the main stream value is small, so that the equations may be linearized. Such an approximation is valid only at the extreme edge of the boundary layer, since as we move into the plate the velocity tends to zero. Nevertheless the method has provided some useful qualitative information about the nature of the flow in the boundary layer. The boundary-layer equation then becomes

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{3.1}$$

neglecting products of v and $(U-u)$.

Write $\zeta=y/\sqrt{(\nu t)}$, $\tau=Ut/x$, and then from a dimensional argument u is a function of ζ, τ only. It will therefore satisfy

$$\frac{\partial^2 u}{\partial \zeta^2}+\frac{\zeta}{2}\frac{\partial u}{\partial \zeta}=(\tau-\tau^2)\frac{\partial u}{\partial \tau}, \tag{3.2}$$

with boundary conditions $u=0$, when $\zeta=0$, and $u\to U$, as $\zeta\to\infty$ or as $\tau\to\infty$. It may be shown that there are no other analytic solutions of the equation. Let us search for a solution of the form

$$u=\int\limits_0^\infty f(s,\tau)\sin\zeta s\,ds. \tag{3.3}$$

There will be a solution of a similar form with $\sin s\zeta$ replaced by $\cos s\zeta$, but this will vanish identically, since $u=0$ when $\zeta=0$. Substituting in the differential equation, we find in the usual way that equation (3.3) satisfies equation (3.2) if

$$[sf\sin s\zeta]_0^\infty=0 \quad\text{and}\quad \tfrac{1}{2}s\frac{\partial f}{\partial s}+\tfrac{1}{2}f+(\tau-\tau^2)\frac{\partial f}{\partial \tau}+s^2 f=0. \tag{3.4}$$

Hence

$$u=\int\limits_0^\infty \frac{e^{-s^2}}{s}g\left(\frac{s^2}{\tau}(\tau-1)\right)\sin s\zeta\,ds,$$

where g is an arbitrary function. As $\tau\to\infty$, $u\to U$ and hence

$$g(+s^2)=\frac{2U}{\pi}e^{s^2}. \tag{3.5}$$

As $\zeta\to\infty$, $u\to U$, which is satisfied unless $\tau\to 0$ simultaneously, so that $\tau\zeta^2$ is constant. It is still satisfied even then if $g(-s^2)=2U/\pi$. Therefore if $\tau<1$,

$$u=\frac{2U}{\pi}\int\limits_0^\infty \frac{e^{-s^2}}{s}\sin s\zeta\,ds=U\operatorname{erf}\frac{\zeta}{2}=U\operatorname{erf}\frac{y}{2\sqrt{(\nu t)}}, \tag{3.6}$$

and if $\tau > 1$

$$u = \frac{2U}{\pi}\int_0^\infty \frac{e^{-s^2/\tau}}{s}\sin s\zeta\,ds = U\,\mathrm{erf}\frac{\zeta\tau^{\frac{1}{2}}}{2} = U\,\mathrm{erf}\left\{\frac{y}{2}\sqrt{\frac{U}{\nu x}}\right\}. \tag{3.7}$$

The significance of this result is more readily seen if we refer the solution to axes fixed in space, and coinciding instantaneously at $t = 0$ with those we have already chosen and moving with the plate.

Distinguishing the new axes by the suffix 1, the solution is

$$u = U\,\mathrm{erf}\frac{y}{2\sqrt{(\nu A)}}, \tag{3.8}$$

where $A = t$ if $x_1 > 0$ and $A = t + x_1/U$ if $0 > x_1 > -Ut$.

The fluid above that part of the plate to the right of the initial position of the leading edge is behaving as though the fluid were of infinite extent and impulsively started from rest at $t = 0$, while that to the left is behaving as though the plate were finite and had always been moving with uniform velocity. This solution satisfies the full equations of motion so long as u is independent of x, and hence may be expected to be correct then. Confirmation of this view will be found in section 5 below. One drawback to this method of approximation is that it cannot be improved directly. If the flow were steady the next approximation would be less accurate than this one, and in our problem the next approximation is discontinuous at $Ut = x$. Such a result is not surprising if the essential singularity we find in section 5 does actually occur. The drag at any point of the plate may be deduced from the value of $\partial u/\partial y$ when $y = 0$. According to this approximation

$$\frac{\partial u}{\partial y} = \frac{U}{\sqrt{(\pi\nu B)}}, \qquad \text{where} \quad B = \min\left(t, \frac{x}{U}\right). \tag{3.9}$$

Equation (3.9) is exact if $Ut < x$, but for large t it is known to overestimate the drag. Further light on the drag variation may be thrown by the momentum integral.

4. The momentum integral

Using Goldstein's notation (**2**) we define

$$\vartheta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy; \qquad \delta_1 = \int_0^\delta \left(1 - \frac{u}{U}\right)dy, \tag{4.1}$$

where δ is the thickness of the boundary layer.

The momentum integral is then

$$\left[\nu\frac{\partial u}{\partial y}\right]_{y=0} = U^2\frac{\partial\vartheta}{\partial x} + U\frac{\partial\delta_1}{\partial t}. \tag{4.2}$$

If now we assume with Lamb (**6**) that

$$u = U \sin \frac{\pi y}{2\delta} \quad (0 < y < \delta), \tag{4.3}$$

then $$\delta_1 = 0{\cdot}363\delta, \qquad \vartheta = 0{\cdot}137\delta, \qquad \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\pi U}{2\delta}, \tag{4.4}$$

and the momentum integral is

$$0{\cdot}363 \frac{\partial \delta^2}{\partial t} + 0{\cdot}137 U \frac{\partial \delta^2}{\partial x} = \pi\nu. \tag{4.5}$$

This is a great simplification of the problem, since we assume that the velocity profile is determined by a scale factor only. The boundary conditions are that $\delta = 0$ when $t = 0$ for $x \geqslant 0$, and when $x = 0$ for $t \geqslant 0$. The solution of this equation is

$$\left.\begin{aligned} \delta &= 2{\cdot}94\sqrt{(\nu t)} && \text{if } Ut \leqslant 2{\cdot}65x \\ \delta &= 4{\cdot}79\sqrt{\left(\frac{\nu x}{U}\right)} && \text{if } Ut \geqslant 2{\cdot}65x \end{aligned}\right\}. \tag{4.6}$$

The corresponding formulae for the drag are that

$$\left.\begin{aligned} \left(\mu \frac{\partial u}{\partial y}\right)_0 &= 0{\cdot}534\rho U \sqrt{\frac{\nu}{t}} && \text{if } Ut \leqslant 2{\cdot}65x \\ \left(\mu \frac{\partial u}{\partial y}\right)_0 &= 0{\cdot}328\rho U \sqrt{\frac{U\nu}{x}} && \text{if } Ut \geqslant 2{\cdot}65x \end{aligned}\right\}. \tag{4.7}$$

These results, unlike those obtained by Rayleigh's method, give good agreement both with Rayleigh's solution for the infinite plate ($t \to 0$), and with Blasius's for the flat plate ($t \to \infty$). The correct numerical values are 0·565 and 0·332 respectively. The method is also known to give good agreement with steady problems in which separation does not occur, and this taken in conjunction with the agreement for small and large t suggests that equation (4.7) is a good approximation to the truth, although the transition is certainly not so abrupt.

Both of these approximate methods require t to go over to x as independent variable at some value of x/Ut, which is an over-simplification of the transition from impulsive motion to steady motion. In the next two sections we shall use the full equations of motion to investigate more closely the way in which this transition takes place.

5. A solution in the neighbourhood of $Ut = x$

When t is small the effect of diffusion far outweighs the effect of convection and the influence of the pressure gradient, and therefore, in particular, the effect of the leading edge. Hence the fluid will move according to Rayleigh's solution for the impulsive motion of an infinite flat plate (**4**).

Since we cannot approximate by means of a power series to the solution for larger t, there must be an essential singularity in the solution for some value of t, and in this section we propose to investigate its existence.

Let us take as independent variables

$$\zeta = \frac{y}{\sqrt{(\nu t)}}, \qquad \tau = \frac{Ut}{x}, \tag{5.1}$$

and write

$$u = U\sqrt{(\nu t)}\,\frac{\partial\psi}{\partial y}, \qquad v = -U\sqrt{(\nu t)}\,\frac{\partial\psi}{\partial x}. \tag{5.2}$$

On dimensional grounds u must be a function of ζ and τ only, and therefore, substituting in the equations of motion (2.1), (2.2), ψ satisfies

$$\frac{\partial^3\psi}{\partial\zeta^3}+\frac{\zeta}{2}\frac{\partial^2\psi}{\partial\zeta^2} = \left(\tau-\tau^2\frac{\partial\psi}{\partial\zeta}\right)\frac{\partial^2\psi}{\partial\zeta\partial\tau}+\tau^2\frac{\partial^2\psi}{\partial\zeta^2}\frac{\partial\psi}{\partial\tau}, \tag{5.3}$$

with boundary conditions

$$\psi = \partial\psi/\partial\zeta = 0 \text{ when } \zeta = 0; \quad \partial\psi/\partial\zeta \to 1 \text{ as } \zeta\to\infty, \quad \text{or as } \tau\to\infty. \tag{5.4}$$

As $t\to 0$, $\tau\to 0$, and $\psi\to\psi_0$ independent of τ, and is given by

$$\frac{\partial\psi_0}{\partial\zeta} = \operatorname{erf}\frac{\zeta}{2}. \tag{5.5}$$

When ζ is large

$$\frac{\partial\psi_0}{\partial\zeta} \sim 1-\frac{2\exp(-\zeta^2/4)}{\zeta\sqrt{\pi}}, \tag{5.6}$$

and has an essential singularity at $\zeta = \infty$.

Equation (5.5) satisfies the differential equation (5.3) and the boundary conditions, except that it does not tend to unity as τ tends to infinity. The obvious next step is to assume an expansion for ψ as a power series in τ whose coefficients are functions of ζ, and whose leading term is ψ_0. However, on substituting in the differential equation it is found that all terms vanish except the leading one, which is independent of τ. This implies that the solution must have an essential singularity at some value of τ. Such a singularity is not necessarily impossible physically and indeed may be highly desirable, if all derivatives with respect to τ vanish there. One example is given by $\psi = 0$, $\tau < 0$, and $\psi = e^{-1/\tau}$, $\tau > 0$, which has a smooth transition at $\tau = 0$. Little is known about the occurrence of essential singularities in differential equations, but they are only expected when the coefficient of a leading term of the differential equation vanishes. Consider the differential equation in the neighbourhood of such a singularity. If we write

$$\psi = \psi_0(\zeta)+\psi_1(\tau,\zeta) \tag{5.7}$$

and neglect ψ_1^2 we have

$$\frac{\partial^3\psi_1}{\partial\zeta^3}+\frac{\zeta}{2}\frac{\partial^2\psi_1}{\partial\zeta^2} = \left(\tau-\tau^2\frac{\partial\psi_0}{\partial\zeta}\right)\frac{\partial^2\psi_1}{\partial\tau\partial\zeta}+\tau^2\frac{\partial^2\psi_0}{\partial\zeta^2}\frac{\partial\psi_1}{\partial\tau}. \tag{5.8}$$

The leading terms of this differential equation are $\partial^3\psi_1/\partial\zeta^3$ and $\partial^2\psi_1/\partial\tau\partial\zeta$, and the coefficient of the second vanishes when $\tau = \tau^2\,\partial\psi_0/\partial\zeta$, i.e. $\tau = 0$ or $\tau \geqslant 1$. However, when $\tau = 0$ the coefficients of the terms arising from the finiteness of the plate are of order τ^2 (these terms come from the differentiation with respect to x in equation (2.2)), while the others are of order τ at most. Hence these terms exert no influence on the first-order solution and it remains independent of τ.

Let us now consider $0 < \tau < 1$, so that the coefficients do not vanish. In this case there is no general proof that an essential singularity cannot occur, but a simpler equation may be used to show that it is unlikely. Consider

$$\frac{\partial^2\phi}{\partial\zeta^2}+\frac{\zeta}{2}\frac{\partial\phi}{\partial\zeta}=\frac{1}{2}\frac{\partial\phi}{\partial\tau}, \tag{5.9}$$

where ϕ vanishes when $\zeta = 0$, when $\tau = \alpha$, and when $\zeta = \infty$. This equation may be obtained from (5.8) by neglecting $\partial\psi_1/\partial\tau$ and simplifying the coefficient of $\partial^2\psi_1/\partial\zeta\partial\tau$. The general solution of this equation for $\tau > \alpha$ may be obtained as in Rayleigh's method (section 3) and is

$$e^{\tau-\alpha}\phi = \int\limits_0^\infty \sin\zeta t\, f(te^{\alpha-\tau})e^{-t^2}\,dt + \int\limits_0^\infty \cos\zeta t\, g(te^{\alpha-\tau})e^{-t^2}\,dt, \tag{5.10}$$

where f and g are only restricted in so far as the integrals are finite. Then since $\phi = 0$ when $\zeta = 0$, $g = 0$, and since $\phi = 0$ when $\tau = \alpha$, $f = 0$ and therefore $\phi \equiv 0$.

Having shown that it is unlikely that an essential singularity will occur before $\tau = 1$, we shall now investigate the possibility of such a solution at $\tau = 1$. It must be emphasized at the outset that the results are more tentative than conclusive. On the other hand, it seems likely that the only possible finite value of τ at which an essential singularity can occur is $\tau = 1$. The difficulty about this equation near $\tau = 1$ is that when ζ is large and $\tau = 1$, the coefficient of

$$\frac{\partial^2\psi_1}{\partial\zeta\partial\tau} \sim \frac{2}{\zeta\sqrt{\pi}}\exp\left(-\frac{\zeta^2}{4}\right),$$

from equation (5.6) and hence has also an essential singularity at $\zeta = \infty$.

We shall now investigate the behaviour of such a solution as

$$(\tau-1)\Big/\left(1-\frac{\partial\psi_0}{\partial\zeta}\right) \to \infty,$$

and then link it up with a solution for more moderate values of

$$(\tau-1)\Big/\left(1-\frac{\partial\psi_0}{\partial\zeta}\right),$$

$\tau-1$ being invariably small. Let $\zeta \to \infty$ in equation (5.8), and keep $\tau-1$ small. The dominating terms of the equation satisfy

$$\frac{\partial^3\psi_1}{\partial\zeta^3}+\frac{\zeta}{2}\frac{\partial^2\psi_1}{\partial\zeta^2} = -(\tau-1)\frac{\partial^2\psi_1}{\partial\tau\partial\zeta}. \tag{5.11}$$

Hence for ζ large the solutions of (5.8) of order $(\tau-1)$ are dominated by

$$(\tau-1)e^{-\zeta^2/4}, \qquad (\tau-1)/\zeta, \quad \text{and} \quad \tau-1, \tag{5.12}$$

while those of order $(\tau-1)^2$ are dominated by

$$(\tau-1)^2\zeta^2e^{-\zeta^2/4}, \qquad (\tau-1)^2/\zeta^3, \quad \text{and} \quad (\tau-1)^2. \tag{5.13}$$

Consequently any solutions we obtain of order $(\tau-1)$ or $(\tau-1)^2$ must be linear combinations of these as $\zeta \to \infty$, and hence their derivatives with respect to ζ must tend to zero as $\zeta \to \infty$ as required for ψ_1. Since the derivatives with respect to τ must also vanish at $\tau = 1$ (so that v will be continuous), we must choose our solution carefully so that the third term of equation (5.12) cannot occur. The behaviour of ψ_1 for moderate values of $(\tau-1)/(1-\partial\psi_0/\partial\zeta)$ with $\tau-1$ still small may now be examined by introducing a new variable b instead of ζ, where

$$b = \frac{8}{\zeta^3\sqrt{\pi}}e^{-\zeta^2/4}. \tag{5.14}$$

Substituting in the differential equation and retaining only the dominating terms for b small, we find that ψ_1 satisfies

$$b^3\frac{\partial^3\psi_1}{\partial b^3}+2b^2\frac{\partial^2\psi_1}{\partial b^2}-\frac{b}{2\log b}\frac{\partial\psi_1}{\partial b} = b\frac{\partial^2\psi_1}{\partial\tau\partial b}\left(b+\frac{\tau-1}{\log b}\right)-b\frac{\partial\psi_1}{\partial\tau}. \tag{5.15}$$

The boundary conditions are that $\psi_1 \to 0$ when $\tau \to 1+$ and also as $b \to \infty$; $\partial\psi_1/\partial\zeta \to 0$ as $b \to \infty$ and ψ_1 is asymptotic to a linear combination of the solutions in equations (5.12) and (5.13) as $\zeta \to \infty$, i.e. $b \to 0$. A major difficulty is the presence of $\log b$ in two of the coefficients of the differential equation (5.15). Since $\log b$ varies more slowly than b, some information may be extracted by setting $\log b = c$, a constant, in these terms. It is precisely this assumption which makes our results rather tentative. Any dominating terms that are found may have to be multiplied by some power of $\log b$, but this should not affect the general conclusions. With this assumption a solution may be found in the form

$$\psi_1 = (\tau-1)^2 f(s), \tag{5.16}$$

where $s = b/(\tau-1)$ and f satisfies

$$s^3\frac{\partial^3 f}{\partial s^3}+\frac{\partial^2 f}{\partial s^2}\left(2s^2+s^3+\frac{s^2}{c}\right)-\frac{\partial f}{\partial s}\left(2s^2+\frac{2s}{c}\right)+2sf = 0. \tag{5.17}$$

The form of equation (5.16) has been used for ψ_1 to eliminate the possibility

that $\psi_1 \to \tau - 1$ as $\zeta \to \infty$ which implies that v would be discontinuous at $\tau = 1$. Near $s = 0$, f is a linear combination of

$$s(-\log s)^{\frac{3}{2}}, \quad (-\log s)^{\frac{3}{2}}, \quad \text{and} \quad 1 \tag{5.18}$$

from equations (5.12) and (5.13). All of these satisfy the boundary conditions on ψ_1 as b and therefore $s \to 0$. On the other hand, when s is large the dominating part of equation (5.17) is

$$s^3 \frac{\partial^3 f}{\partial s^3} + s^3 \frac{\partial^2 f}{\partial s^2} - 2s^2 \frac{\partial f}{\partial s} + 2sf = 0, \tag{5.19}$$

the solutions of which are $f = s$, $f = s^2$ and

$$f \sim \frac{e^{-s}}{s^2} \quad \text{or} \quad \psi_1 \sim \frac{(\tau-1)^4}{b^2} \exp\left(-\frac{b}{(\tau-1)}\right). \tag{5.20}$$

Of these only the latter is acceptable as $s \to \infty$, since f tends to zero exponentially there. Hence we have found a solution satisfying all our conditions; for as $\tau \to 1+$ for fixed ζ, $s \to \infty$ and $f \to 0$; as $\zeta \to 0$, $s \to \infty$ and again $f \to 0$, so that u, v are zero at the wall; as $\zeta \to \infty$ for fixed τ, $s \to 0$ and all of the three possibilities (5.18) satisfy the conditions that $\partial\psi_1/\partial\zeta \to 0$ and $\partial\psi_1/\partial\tau$ be continuous there. Its range of validity may now be improved by repeated substitution into the differential equation and the arbitrary constant may be determined by the boundary condition as $\tau \to \infty$.

Although we have found a solution with an essential singularity at $\tau = 1$, there is still a possibility that an essential singularity may not be developed until $Ut = \alpha x$, where $\alpha > 1$, and that ψ is independent of x if $Ut/x < \alpha$. We may only say that in this case the methods which have proved successful in the rest of this paper lead to a contradiction. Hence although the possibility cannot be ruled out, its occurrence seems unlikely.

6. The nature of the solution at large times

Let us take as independent variables

$$\xi = y\sqrt{\left(\frac{U}{\nu x}\right)}, \qquad \eta = \frac{x}{Ut} \tag{6.1}$$

and write
$$u = \frac{\partial}{\partial y}\{\phi\sqrt{(\nu x U)}\}, \qquad v = -\frac{\partial}{\partial x}\{\phi\sqrt{(\nu x U)}\}. \tag{6.2}$$

On dimensional grounds u must be a function of ξ and ζ only, and therefore, substituting in the equations of motion (2.1), (2.2), ϕ satisfies

$$\frac{\partial^3\phi}{\partial\xi^3} + \frac{\phi}{2}\frac{\partial^2\phi}{\partial\xi^2} = \left(\eta\frac{\partial\phi}{\partial\xi} - \eta^2\right)\frac{\partial^2\phi}{\partial\xi\partial\eta} - \eta^2\frac{\partial^2\phi}{\partial\xi^2}\frac{\partial\phi}{\partial\eta} \tag{6.3}$$

with boundary conditions $\phi = \partial\phi/\partial\xi = 0$ when $\xi = 0$; and $\partial\phi/\partial\xi \to 1$ both as $\xi \to \infty$ and as $\eta \to \infty$. As $\eta \to 0$, $t \to \infty$ and the motion becomes steady, in which case the correct solution is Blasius's, for which ϕ is a function of ξ

only. Denoting this solution by ϕ_0, then ϕ_0 satisfies all the boundary conditions except that it does not tend to unity as η tends to infinity. Again, it is not possible to expand ϕ as a power series in η whose coefficients are functions of ξ, since then the term arising from the time variation of u is of smaller order than that of the others. Again this suggests that we look for an essential singularity, and the work of the previous section suggests that we look for one at $\eta = 0$, since it is unlikely to arise at any other value. It must be borne in mind in this section that we are now trying to introduce a time variation into the solution, whereas previously we had been trying to introduce a variation in x. Such a singularity is likely at $\eta = 0$ for the coefficient of $\partial^2\phi/\partial\xi\partial\eta$, $\eta(\partial\phi/\partial\xi)-\eta^2$ has a simple zero at $\eta = 0$, except that if $\xi = 0$ as well, it has a double zero.

Let us consider a possible form for such a singularity at $\eta = 0$, bearing in mind that, as in all boundary-layer problems, we cannot be absolutely certain that our solution will occur in practice, since no uniqueness theorem has been proved. Indeed there are an infinite number of possibilities even for the form we find. However, we can be much more confident of the form of such a solution than we could be in the previous case, because we shall not have had to assume that any slowly varying function, such as $\log b$, is constant, and we have been able to obtain an explicit form for the dominating term.

Let
$$\phi = \phi_0(\xi)+f(\xi,\eta), \tag{6.4}$$

where $f \to 0$ as $\eta \to 0$ and as $\xi \to 0$; $\partial f/\partial\xi \to 0$ as $\xi \to 0$ and as $\xi \to \infty$. Further introduce a new variable θ instead of ξ defined by $\xi = \theta\eta$. Then if η is small f satisfies

$$\frac{1}{\eta^3}\frac{\partial^3 f}{\partial\theta^3}+\tfrac{1}{2}\phi_0\frac{\partial^2 f}{\eta^2\partial\theta^2}+\tfrac{1}{2}f\frac{\partial^2\phi_0}{\eta^2\partial\theta^2}$$
$$= \left(\frac{\partial\phi_0}{\partial\xi}-\eta\right)\left(\frac{\partial^2 f}{\partial\theta\partial\eta}-\frac{1}{\eta}\frac{\partial f}{\partial\theta}-\frac{\theta}{\eta}\frac{\partial^2 f}{\partial\theta^2}\right)-\frac{\partial^2\phi_0}{\partial\xi^2}\left(\eta\frac{\partial f}{\partial\eta}-\theta\frac{\partial f}{\partial\theta}\right), \tag{6.5}$$

neglecting squares of f. Consider the following function

$$f(\xi,\eta) = \frac{1}{\theta}\frac{\partial\phi_0}{\partial\xi}g(\theta,\eta)\exp\left(-\frac{a^3}{9\eta^3}\right), \tag{6.6}$$

which has an essential singularity at $\eta = 0$, if $a > 0$. It will appear later that as $\theta \to \infty$, $g \sim \theta$. Hence although when η is small and θ is moderate

$$\frac{1}{\theta}\frac{\partial\phi_0}{\partial\xi} \sim \eta,$$

we still need this term, because without it f would not tend to a finite limit

as $\xi \to \infty$. For moderate values of θ, since η is small, $\partial\phi_0/\partial\xi = 0{\cdot}332\xi$, and therefore g satisfies

$$\frac{\partial^3 g}{\partial\theta^3} = \frac{a^3}{3}(\lambda\theta-1)\frac{\partial g}{\partial\theta} - \frac{\lambda a^3}{3} g + O(\eta^3), \tag{6.7}$$

where $\lambda = 0{\cdot}332$. From the boundary conditions on f, $\partial^3 g/\partial\theta^3$ must also vanish when $\theta = 0$. The appropriate solution is

$$\frac{\partial^2 g}{\partial\theta^2} \propto (\lambda\theta-1)^{\frac{1}{2}} K_{\frac{1}{3}}\left[\frac{2}{\lambda}\left(\frac{a(\lambda\theta-1)}{3}\right)^{\frac{3}{2}}\right], \tag{6.8}$$

where $K_{\frac{1}{3}}$ denotes a Bessel function of order $\frac{1}{3}$. This form of the solution is also valid when $\lambda\theta < 1$, because it is merely a power series, with integral indices, in $\lambda\theta-1$, but it may also be expressed in terms of the Bessel functions $J_{\frac{1}{3}}$ and $J_{-\frac{1}{3}}$, (**3**). We may now determine a from the condition that $\partial^3 g/\partial\theta^3 = 0$ when $\theta = 0$. For expressing $\partial^2 g/\partial\theta^2$ in terms of $J_{\frac{1}{3}}$ and $J_{-\frac{1}{3}}$, if $\partial^3 g/\partial\theta^3 = 0$ at $\theta = 0$, then

$$\frac{d}{dy}[y^{\frac{1}{3}}J_{\frac{1}{3}}(y) + y^{\frac{1}{3}}J_{-\frac{1}{3}}(y)] = 0, \qquad \text{where} \quad y = \frac{2a^{\frac{3}{2}}}{3\lambda\sqrt{3}}. \tag{6.9}$$

Since these two functions are oscillatory when $y > 0$, there are an infinite number of roots of this equation of which the least is $y = 0{\cdot}613$ or $a = 0{\cdot}648$. If we considered any other root, g would be oscillatory when $\lambda\theta < 1$, which is not very likely to occur. When θ is large

$$\frac{\partial^2 g}{\partial\theta^2} \sim \frac{1}{\theta^{\frac{1}{4}}}\exp\left[-2\lambda^{\frac{1}{2}}\left(\frac{a\theta}{3}\right)^{\frac{3}{2}}\right]. \tag{6.10}$$

g and $\partial g/\partial\theta$ may be obtained by integration and using the boundary conditions when $\theta = 0$. From equations (6.10) and (6.7) it follows that

$$\frac{\partial g}{\partial\theta} \sim \lambda A + o(1), \tag{6.11}$$

where A is a function of η only, and

$$g \sim (\lambda\theta-1)A + o(1). \tag{6.12}$$

Hence for moderate values of ξ, and η very small,

$$f \sim A\frac{\partial\phi_0}{\partial\xi}\exp\left(-\frac{a^3}{9\eta^3}\right), \tag{6.13}$$

and this may be verified to satisfy the differential equation, and tend to a finite limit as $\xi \to \infty$ so that the boundary condition there is satisfied.

We have therefore obtained a possible form for the solution near $\eta = 0$ with an essential singularity at $\eta = 0$. The boundary condition as $\eta \to \infty$ may presumably be used to determine A. According to this solution the influence of η on the velocity parallel to the plate is greatest near the plate, dying out like $\partial^2\phi_0/\partial\xi^2$ as $\xi \to \infty$ and like $\exp(-0{\cdot}03/\eta^3)$ as $\eta \to 0$.

There is a distinction between the two essential singularities we have obtained. In this one the singularity occurs everywhere on the line $\eta = 0$ independently of how it was approached, while near $Ut = x$ it has an essential singularity if $Ut \to x+$ for fixed ζ but not for fixed s, no boundary conditions being, of course, violated. The reason for this difference in the behaviour of the two singularities is in the character of the coefficients of $\partial^2\phi/\partial\xi\partial\eta$ and of $\partial^2\psi/\partial\zeta\partial\tau$ at the critical values of Ut/x. The first has a simple zero when $\eta = 0$ except that, if in addition $\xi = 0$, it has a double zero. The second is zero only when $\zeta = \infty$, having an essential singularity there. If the coefficient has a simple zero at the critical value for all y, it is sometimes, but not always, possible to find a solution of the differential equation in question satisfying given boundary conditions. The Rayleigh method, discussed in section 3, is an example of the former possibility.

The results so far may be summarized as follows. After the plate has been given its impulsive velocity, the velocity in the boundary layer is independent of x until $Ut = x$. Then, starting at the edge of the boundary layer with an essential singularity and moving in towards the plate, the x-coordinate begins to affect the solution. Ultimately it becomes the most important influence, the change over being about complete when $Ut = 2{\cdot}65x$ and the influence of t dies out like $\exp(-0{\cdot}03U^3t^3/x^3)$. There may well be other solutions of a diffetent kind, but we have been unable to find them.

7. The effect of compressibility

The mathematical theory of the boundary layer is complicated by compressibility effects in two main ways. In the first place, the physical quantities, viscosity and diffusivity, which were assumed to be constant in the incompressible theory, must now vary in a prescribed way with the temperature. Secondly, four equations, inter-relating the temperature, the density, and the components of the fluid velocity along and perpendicular to the boundary, have to be integrated, as compared with one only in the incompressible case. It is abundantly clear, in fact, that electronic calculating machines are necessary to give a complete picture of the flow.

The literature on steady compressible boundary layers is quite considerable. It may be shown quite easily that when the motion is steady the number of dependent variables can be reduced to two, namely the stream function and the temperature. To effect further simplification, special assumptions must be made as to the nature of the viscosity and the diffusivity and also as to the thermal behaviour of the boundary. For example, if we assume that the ratio of the diffusivity and the viscosity is unity, and that the boundary is thermally insulating, then the temperature may be expressed in terms of the stream function, so that the problem

is reduced to the solution of a single partial differential equation of the same character as that for incompressible flow. If in addition to these two assumptions the viscosity is assumed to be proportional to the absolute temperature, then a suitable transform correlates the compressible boundary layer to an incompressible one with a different main stream velocity.

At room temperatures it is known empirically that

$$\mu \propto T^{8/9}, \tag{7.1}$$

where T is the absolute temperature, and also that the Prandtl number

$$\sigma = \mu c_p/k = 0{\cdot}715, \tag{7.2}$$

where k is the thermal conductivity, and c_p the specific heat at constant pressure. The mathematical theory of the steady compressible boundary layer mentioned above can therefore be expected to be qualitatively correct and to exhibit the main features of the flow. The agreement with the more exact theory in certain problems is surprisingly good.

On the other hand, the theory of the unsteady compressible boundary layer is by no means so well developed, chiefly on account of the complexity of the equation of continuity consequent on the introduction of the time derivative. It will be shown here that a suitable transformation can considerably simplify the equations of motion, reducing them to two simultaneous partial differential equations, for the stream function and the absolute temperature. In the particular problem in which we are interested there is no pressure gradient in the main stream and a complete correlation exists between the compressible and incompressible boundary layers.

The equations of motion appropriate to a boundary layer in the neighbourhood of a plane wall are as follows:

(*a*) the equation of continuity,

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0; \tag{7.3}$$

(*b*) the Prandtl dynamical equations

$$\rho\left(\frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial x}+v\,\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\,\frac{\partial u}{\partial y}\right), \tag{7.4}$$

$$0 = -\frac{\partial p}{\partial y}; \tag{7.5}$$

and (*c*) the equation of energy

$$\rho c_p\left[\frac{\partial T}{\partial t}+u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial y}\right]-\frac{\partial p}{\partial t}-u\,\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{7.6}$$

where x, y are the coordinates of any point measured along and perpendicular to the wall, and u, v are the corresponding components of the fluid velocity. In addition there is an equation of state

$$p = R\rho T, \tag{7.7}$$

where R is a constant, and the boundary conditions are

$$u = v = 0 \quad \text{when } y = 0; \qquad u \to U_1,\ T \to T_1 \text{ as } y \to \infty, \tag{7.8}$$

with a condition on T at the wall depending on its thermal character. These five equations and five boundary conditions are sufficient to specify the flow. If the flow were steady, $\partial\rho/\partial t$ would vanish, and a stream function can be defined at once from the equation of continuity. For unsteady flows, however, we have to use two functions ψ, Y and specify ρ, ρu, ρv in terms of them. If we set

$$\rho u = \rho_0\left(\frac{\partial\psi}{\partial y}\right)_{x,t}; \qquad \rho = \rho_0\left(\frac{\partial Y}{\partial y}\right)_{x,t}, \tag{7.9}$$

then it follows that

$$\rho v = -\rho_0\left(\frac{\partial\psi}{\partial x}\right)_{y,t} - \rho_0\left(\frac{\partial Y}{\partial t}\right)_{x,y}. \tag{7.10}$$

The relation between ρ and Y is equivalent to introducing a new coordinate Y, instead of y, defined by

$$Y = \int_0^y \rho\, dy/\rho_0. \tag{7.11}$$

The suffix 0 denotes some standard state depending on the problem under consideration, and the suffix 1 denotes conditions in the main stream $(y \to \infty)$. In terms of x, Y, t

$$u = \left(\frac{\partial\psi}{\partial Y}\right)_{x,t}, \tag{7.12}$$

and

$$\left(\frac{\partial\psi}{\partial x}\right)_{y,t} = \left(\frac{\partial\psi}{\partial x}\right)_{Y,t} + \left(\frac{\partial\psi}{\partial Y}\right)_{x,t}\left(\frac{\partial Y}{\partial x}\right)_{y,t}, \tag{7.13}$$

so that

$$\frac{\rho v}{\rho_0} = -\left(\frac{\partial\psi}{\partial x}\right)_{Y,t} - \left(\frac{\partial\psi}{\partial Y}\right)_{x,t}\left(\frac{\partial Y}{\partial x}\right)_{y,t} - \left(\frac{\partial Y}{\partial t}\right)_{x,y}. \tag{7.14}$$

Hence we may show, after a little reduction, that

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2\psi}{\partial Y\partial t} + \frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial x\partial Y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial Y^2}. \tag{7.15}$$

Further,

$$\mu\frac{\partial u}{\partial y} = \frac{\mu\rho}{\rho_0}\frac{\partial u}{\partial Y},$$

so that

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \frac{p_1\rho}{p_0\rho_0}\frac{\partial}{\partial Y}\left(\frac{\mu T_0}{T}\frac{\partial^2\psi}{\partial Y^2}\right), \tag{7.16}$$

and moreover

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{T}{T_1}\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}, \tag{7.17}$$

since from equation (7.5) p is independent of y. Hence the equation of motion is

$$\frac{\partial^2\psi}{\partial Y\partial t} + \frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial x\partial Y} - \frac{\partial^2\psi}{\partial Y^2}\frac{\partial\psi}{\partial x} = -\frac{T}{T_1}\frac{1}{\rho_1}\frac{\partial p_1}{\partial x} + \frac{\nu_0 p_1}{p_0}\frac{\partial}{\partial Y}\left(\frac{\mu T_0}{\mu_0 T}\frac{\partial^2\psi}{\partial Y^2}\right), \tag{7.18}$$

and has been reduced to a similar form to that for the corresponding steady flow. It is interesting to observe that the significant transformation, equation (7.11), is the same as in steady compressible flow, and that the important viscosity parameter is not μ but μ/T, so that those fluids in which μ is proportional to the absolute temperature are easier to handle than those in which it is constant. To complete the transformation, the energy equation must also be reduced, and in a similar way we get

$$Jc_p\left[\frac{\partial T}{\partial t}+\frac{\partial\psi}{\partial Y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial Y}\right]-\frac{T}{\rho_1 T_1}\left[\frac{\partial p_1}{\partial t}+\frac{\partial\psi}{\partial Y}\frac{\partial p_1}{\partial x}\right]$$
$$=\frac{Jc_p\nu_0 p_1}{p_0}\frac{\partial}{\partial Y}\left(\frac{\mu T_0}{\mu_0 T\sigma}\frac{\partial T}{\partial Y}\right)+\frac{p_1\mu T_0}{\rho_0 p_0 T}\left(\frac{\partial^2\psi}{\partial Y^2}\right)^2. \quad (7.19)$$

The equations of motion have thus been reduced to a pair of simultaneous equations in ψ and T, analogously to those for a steady boundary layer. Without further assumptions it does not appear possible to effect any further real simplification. This is not surprising, since if the fluid is compressible the unsteady motion in the main stream where the viscosity may be neglected is imperfectly understood. For the problem in which we are interested, the pressure in the main stream is constant and the main features of the flow may be exhibited, if we make use of the following assumptions which have already proved useful in steady problems:

(*a*) the viscosity varies as the absolute temperature,

(*b*) $\sigma = 1$,

(*c*) the boundary is thermally insulating.

The dynamical equation of motion is now the same whether the fluid is compressible or not and hence the two solutions of the problem are formally the same. If the fluid is compressible, however, the solution is only completed when we have transformed back from Y to y. For this purpose we need the energy integral given by Crocco (**5**), which is

$$T = T_1\left\{1+\frac{\gamma-1}{2\gamma}\frac{\rho_1}{p_1}(U_1^2-u^2)\right\}. \quad (7.20)$$

This satisfies the boundary conditions as $Y \to \infty$ and as $Y \to 0$, and hence

$$y = \int_0^Y \left[1+\frac{\gamma-1}{2\gamma}\frac{\rho_1}{p_1}(U_1^2-u^2)\right]dY. \quad (7.21)$$

Thus one effect of compressibility is to thicken the boundary layer, because Y may be taken to represent the incompressible flow and y the compressible flow. At corresponding states of the fluids $y > Y$. If $Ut < x$

$$u = \operatorname{erf}\frac{Y}{2\sqrt{(\nu_0 t)}}, \quad (7.22)$$

and hence when y is large

$$y = Y+(\gamma-1)M^2\sqrt{(2\nu_0 t)}, \tag{7.23}$$

writing M for the Mach number of the main stream velocity. Moreover for large Y,

$$\psi = UY-2U\sqrt{(\nu_0 t/\pi)}, \tag{7.24}$$

and hence the outflow v into the main stream is

$$v = \frac{\gamma-1}{2}M^2\sqrt{\left(\frac{2\nu_0}{t}\right)}. \tag{7.25}$$

If this velocity normal to the plate is applied as a boundary condition in the main stream where viscosity may be neglected, a compression wave with infinitely sharp front would be found to be radiated outwards. Since this wave is inversely proportional to the square root of the Reynolds number, its effect on the boundary layer would be small were it not for the singularity when $t = 0$. For an investigation of the interaction of boundary layer and compression wave we must consider the full equations of motion, which is beyond the scope of this paper.

When $Ut > x$ the full solution is not known, but the linearized equation (section 3) then gives an outflow of

$$v = \left[1+\frac{\gamma-1}{2}M^2\sqrt{(2\pi)}\right]\sqrt{\left(\frac{U\nu_0}{\pi x}\right)}, \tag{7.26}$$

implying a second compression wave when $Ut = x$. The discussion of the boundary-layer equations suggests that the outflow is not discontinuous at $Ut = x$, but that it gradually rises up to its ultimate value as $Ut/x \to \infty$.

The author would like to express his thanks to Professor Howarth for stimulating discussions on this paper.

REFERENCES

1. S. Goldstein, *Modern Developments in Fluid Dynamics* (Oxford, 1938), vol. 1 pp. 181–86.
2. Ibid. p. 156.
3. G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1944), 188–90.
4. Lord Rayleigh, *Phil. Mag.* (6), **21** (1911), 697–711.
5. L. Crocco, *Monogr. Sci. Aeronaut.*, No. 3 (1946).
6. H. Lamb, *Hydrodynamics* (Cambridge, 1932), 686.

THE APPLICATION OF RELAXATION METHODS TO THE SOLUTION OF DIFFERENTIAL EQUATIONS IN THREE DIMENSIONS

I. BOUNDARY VALUE POTENTIAL PROBLEMS

By D. N. DE G. ALLEN *and* S. C. R. DENNIS
(*Imperial College, London*)

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SUMMARY

The extension of the use of the relaxation method from two to three dimensions has for a long time been desirable; the use of a three-dimensional network has been thought to be impractical because it has been supposed that it must require the superposition of a set of plane nets each drawn on a separate sheet. An alternative method was proposed by Tranter (**1**) in which he attempted to replace one three-dimensional problem by a set of two-dimensional problems; his method is satisfactory, but in an extremely limited capacity in that it only solves Laplace's (or Poisson's) equation inside a volume which is bounded by a cylinder (of any cross-section) with plane ends normal to the generators. In this paper we show that provided the work is properly set out, then there is no particular difficulty in the use of a three-dimensional network; as illustrative examples solutions have been found for the steady temperature-distribution problem used by Tranter in describing his method (for comparison); and also for the electric potential-distribution inside a quadrant electrometer.

1. THE use of relaxation methods to solve a Poisson-type equation in two dimensions is now familiar† and does not need to be described again. Some of the statements and equations required in this description of the extension of the methods into three dimensions will be recognized as being the counterparts of similar results already proved in two dimensions; it is not necessary to prove them again or to elaborate on their significance. We therefore only describe in detail here those aspects of the method which are either peculiar to three-dimensional problems or have not previously been specifically noted in two dimensions.

2. We propose to show how a numerical solution in three dimensions of the equation

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2} = W(x,y,z) \tag{1}$$

can be found by determining, to any required order of accuracy, values of the required quantity, w, at the joints of a uniform cubical lattice which extends throughout the volume of integration; W is supposed to be a

† A detailed description has been given by Southwell (**2**).

[Quart. Journ. Mech. and Applied Math., Vol. IV, Pt. 2 (1951)]

function of x, y, and z known in so far as a numerical value can be assigned to it at all the joints of the lattice. The method consists essentially of replacing the differential equation (1) by a set of finite-difference approximations which relate the value of w at a typical joint of the lattice with the values at the six surrounding joints. Fig. 1 represents such a typical joint, 0, and the six adjacent joints are denoted by 1, 2, 3, 4, 5, and 6 as

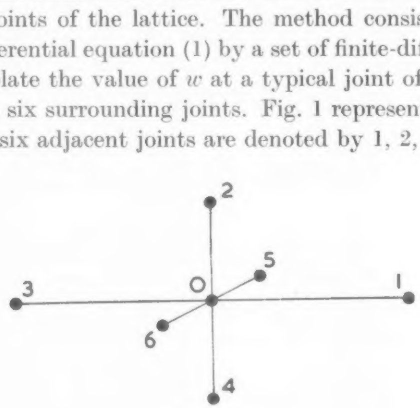

FIG. 1.

shown; when the mesh-distance between neighbouring joints is taken to be h, the finite-difference replacement for the equation (1), holding at the joint 0, is known to be

$$w_1+w_2+w_3+w_4+w_5+w_6-6w_0-h^2W_0 = 0, \tag{2}$$

where suffixes denote values of w and W at correspondingly numbered joints. In the process of a relaxation computation the values of w at the joints change gradually from some initial (and arbitrarily chosen) values and eventually converge to the finally accepted values which constitute the solution. When this convergence is completed the equation (2) must

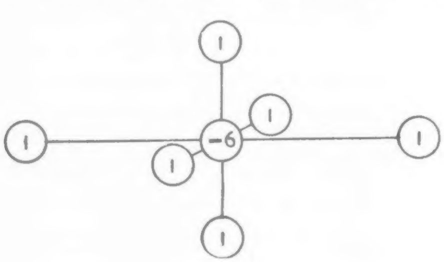

FIG. 2.

be satisfied at each joint of the lattice as nearly as that is possible, depending on the number of significant figures which are used; at any stage in the process of the computation, before the accepted solution has been reached, the equation (2) will not necessarily be satisfied at any joint by the w-values obtaining at that stage. The amounts by which that equation is not then

satisfied are measured at each node by a residual defined by the expression

$$F_0 \equiv w_1+w_2+w_3+w_4+w_5+w_6-6w_0-h^2W_0. \tag{3}$$

The aim of the relaxation method is eventually to make every residual as near to zero as possible by continual adjustment of the w-values. The work is systematically and easily accomplished by the use of a relaxation pattern given in Fig. 2, which shows the changes occasioned in the values of the residuals when a unit increment is added to w at a typical joint of the lattice.

3. The major difficulty which has hitherto prevented three-dimensional relaxation is that involved in finding a satisfactory technique of recording on paper a calculation which in reality should take place on a three-dimensional lattice. Tranter avoids the difficulty by, in effect, using Fourier expansions in one (say z) of the coordinates. His solution w is then determined in the infinite series form,

$$w = \frac{2}{\pi}\sum_{n=1}^{\infty} \bar{w}(n)\sin nz,$$

where $\bar{w}(n)$ is a function of x and y only which can be found by a two-dimensional relaxation. He thus reduces the one three-dimensional problem to a number of two-dimensional problems—to find $\bar{w}(1)$, $\bar{w}(2)$, $\bar{w}(3)$, etc.; but this can be done only at the expense of restricting the shape of the volume of integration to be cylindrical and by requiring the boundary condition on the plane (z = constant) ends to be either given values of w or of the normal gradient of w.

4. To illustrate how a three-dimensional lattice may be used we take Tranter's example—to determine the steady temperature distribution inside a uniform cube when one face is maintained at a constant temperature (say 1,000) and all the other faces are kept at zero temperature. The equation governing the temperature, w, throughout the cube is Laplace's, i.e. equation (1) with $W = 0$, so that the formula (3) which defines residuals at each joint of the lattice reduces simply to

$$F_0 = w_1+w_2+w_3+w_4+w_5+w_6-6w_0. \tag{4}$$

It may be noticed that we do not need to specify any particular absolute size of the cube. If its side is of length L, then its six faces may be taken to be the planes x, y, $z = 0$ or L and we will suppose that the face $y = L$ is the one on which w is kept at 1,000. The solution was obtained initially on a coarse lattice with each edge of the cube containing four mesh-lengths ($h = L/4$). This complete lattice covering the whole cube is shown in

Fig. 3: boundary values of w are specified at the joints of the lattice in the faces of the cube, and twenty-seven internal joints can be counted; from each of these joints six lattice-lines radiate in the axial directions, and once initial (guessed) values of w have been assigned at these joints, residuals can be calculated by substitution in the formula (4). They may then be relaxed by use of the usual methods of relaxation and, in particular,

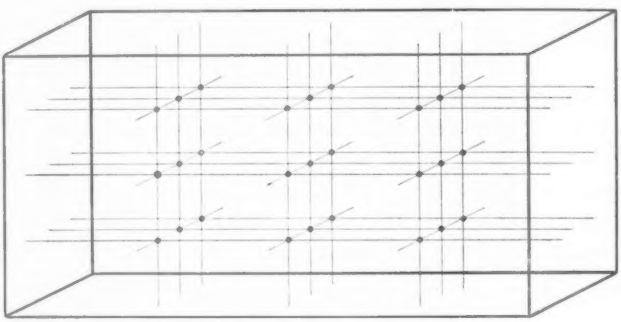

FIG. 3.

of the pattern of Fig. 2, which has been drawn to correspond with Fig. 3. Fig. 3 itself is given primarily to illustrate how a (typical) three-dimensional lattice should be drawn in order that a relaxation computation may be carried out on it; it can be seen that it allows at each joint of two columns of values being written down, one consisting of increments added to the w-value there and the other of successive values of the residual. It is necessary in drawing the lattice to allow only one set of plane sections to overlap one another. To achieve this and at the same time to compress the diagram into as small a space as possible necessitates considerable relative distortion of the mesh-lengths in the three axial directions. *It is this 'isometric projection' of the three-dimensional lattice which enables the work to be carried out on a single sheet of paper.*

The three dimensions of the region shown in Fig. 3 are apparent in spite of the distortion already mentioned; this distortion is in no way troublesome because in the actual computations of the relaxation process made on it, *the lattice is never thought of in three dimensions but instead as a two-dimensional network with nodes at all points marked with dots*; all other (unmarked) points where strings cross are not regarded as nodes of the net. This view of the actual working diagram greatly facilitates the relaxation: in particular a block-displacement can be applied for any chosen group of nodes and its effects on the residuals immediately recorded by the usual and simple two-dimensional technique of 'counting strings'.

5. In this example there is obvious symmetry in both of the planes $x = L/2$ and $z = L/2$, so only one-quarter of the whole cube need actually be used in the computation and in the presentation of results; these are given in Figs. 4 and 5. Fig. 4 records values of w, at the joints, as found on the coarse lattice ($h = L/4$). Fig. 5 similarly records values of w found at the joints of a finer lattice ($h = L/8$). Although, of course, the actual

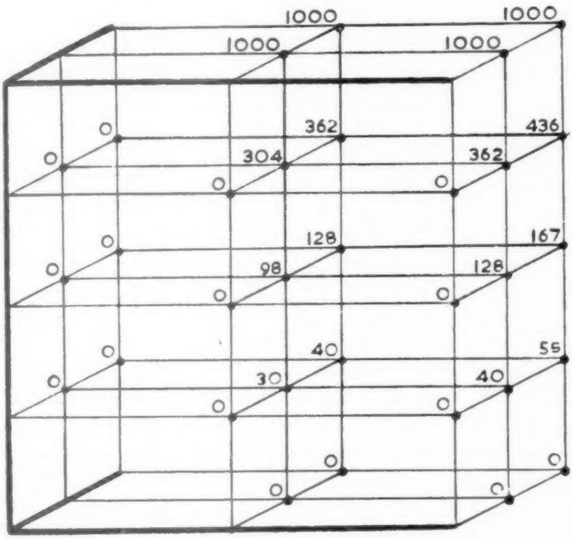

FIG. 4.

process of relaxation on a three-dimensional lattice is rather more complicated than it is on a two-dimensional net, it is nevertheless no more laborious to arrive at a solution and indeed, with a little experience, *convergence is probably more rapid in three dimensions than in two for a comparable number of joints and nodes respectively.* The reason for this is that the ratio of boundary to internal joints in three dimensions is greater than the corresponding ratio of boundary to internal nodes in two dimensions; consequently it is easier in three dimensions to liquidate initial residuals by the relaxation process of moving them to the boundary, where, in a boundary value problem at least, they are not required to vanish. Thus in the finer lattice of the example given in this section there are 243 internal joints at any of which there may initially be non-zero residuals to be removed; but none of these joints are more than *four* mesh-distances from some part of the boundary. A corresponding net in two dimensions would cover a square with sixteen mesh-lengths in each

side (this net would include 225 internal nodes); and the individual nodes of such a square net are anything up to *eight* mesh-lengths from the nearest part of the boundary.

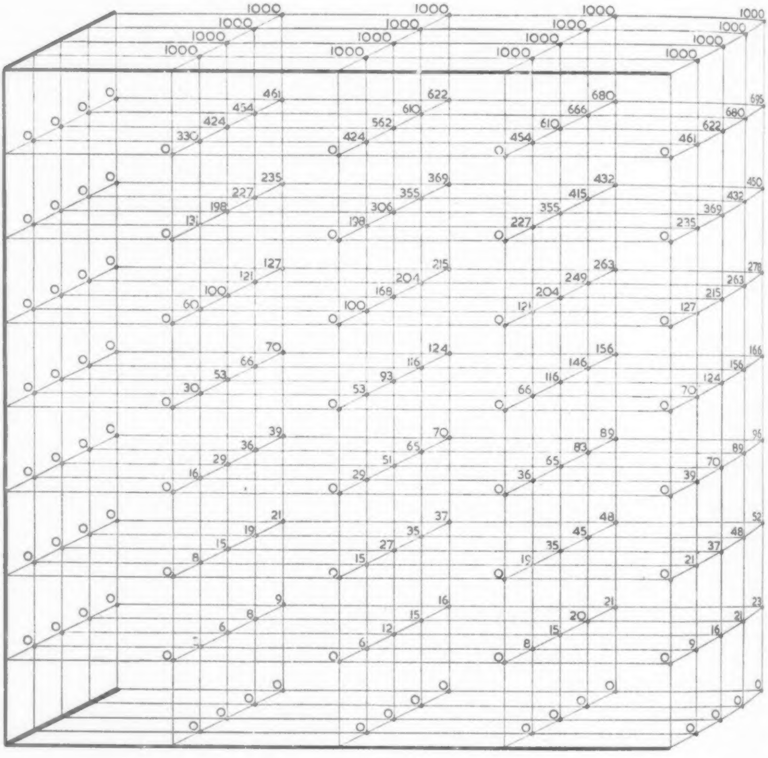

FIG. 5.

6. An exact analytical solution in the form of a double infinite series has been found for this problem by Carslaw (**3**) and is quoted by Tranter. Calculation from it of the values of w at the joints of the lattice used in the relaxation solution enable a test for accuracy to be made in this first example to be solved by three-dimensional relaxation. It is found that for the coarse lattice of Fig. 4 the maximum error in the w-values is 22, or 2·2 per cent. of the greatest w-value. For the finer lattice of Fig. 5 this maximum error is reduced to 1·7 per cent. No particularly useful purpose would be served by making any comparison with Tranter's results, and this is not done since it would entail considerable extra computation: for

he gives solutions for the functions $\bar{w}(1)$, etc. (cf. section 3), but uses them to find w itself at only one† point.

7. In general the bounding surface of the volume of integration may be curved and then some of the strings of the lattice radiating from joints just inside the boundary would be cut short by it. Fig. 6 illustrates a possible

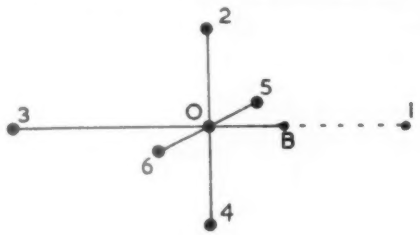

FIG. 6.

case where the string OB actually meets the boundary at the point B. The point 1 then would lie just outside the volume of integration and any value w_1 which may be attached to it is a fictitious value; the residual at the point 0 is still given by the formula (4), but in it is contained this fictitious w_1 which must be eliminated before the residual F_0 can be evaluated. The necessary elimination can be performed by making use of the fact that the boundary condition provides a known value w_B. If OB is a fraction ξ of the complete mesh-length h, then it is easily shown that, with an error of $O(h^3)$,

$$w_1 = \frac{2}{\xi(1+\xi)} w_B + \frac{1-\xi}{1+\xi} w_3 - \frac{2(1-\xi)}{\xi} w_0. \tag{5}$$

For with 0 as origin w may be expanded in powers of x along the line 301:

$$w = w_0 + px + qx^2 + \dots,$$

where $p = (\partial w/\partial x)_0$ and $q = (\partial^2 w/\partial x^2)_0/2!$. Putting x equal in turn to ξh, h, $-h$ in this series, we get

$$w_B = w_0 + p\xi h + q\xi^2 h^2,$$
$$w_1 = w_0 + ph + qh^2,$$
$$w_3 = w_0 - ph + qh^2,$$

with neglect of terms containing h^3 and higher powers of h. Elimination of p and q from these three equations yields the relation (5). When this

† i.e. at the centre of the cube; it may be remarked that the exact value there is obviously 1000/6 since the effect of maintaining one face of the cube at a temperature of 1,000 is by symmetry one-sixth of the effect of so maintaining all six faces. Any numerical method which makes use of this symmetry must yield a correct value at that point; and testing such a solution there affords no check of accuracy whatsoever.

relation is used to substitute for the fictitious w_1 in the residual formula (4) we find that it becomes†

$$F_0 \equiv w_2+w_4+w_5+w_6+\frac{2}{1+\xi}w_3+\frac{2}{\xi(1+\xi)}w_B-\left(4+\frac{2}{\xi}\right)w_0. \tag{6}$$

A short string radiating from a joint 0 in any of the other five directions can be dealt with in an exactly similar manner; and an obvious combination of two or more of these cases can be made when more than one of the strings radiating from a joint are cut short by the boundary.

8. A modified residual formula is also used at joints which lie in a plane of symmetry. Thus if 0 typifies such a joint and (Fig. 1) the plane of symmetry is the plane 012, it is not necessary to record the field on both sides of this plane: the joint 6 (say) would not be shown since it is simply the reflection of the joint 5, consequently in the residual formula (4) the value w_6 is in effect fictitious; it is eliminated by means of the relation

$$w_6 = w_5, \tag{7}$$

so that the residual, F_0, is given by

$$F_0 \equiv w_1+w_2+w_3+w_4+2w_5-6w_0. \tag{8}$$

It is worth mentioning that one can satisfy a normal gradient boundary-condition at a plane boundary by this same treatment; for a plane of symmetry is in effect a boundary (of the volume which is recorded) with a condition imposed there that $\partial w/\partial \nu = 0$ (ν being the direction normal to the plane).‡ If, more generally, we require to impose the condition $\partial w/\partial \nu = N(x,y,z)$, then the only difference is that (7) is replaced by

$$w_6 = w_5+2hN_0,$$

and (8) by $$F_0 \equiv w_1+w_2+w_3+w_4+2w_5-6w_0+2hN_0.$$

9. As our second worked example we take one which includes a curved boundary and therefore a use of the irregular star residual-formula (6), and for which also Tranter's method is not directly applicable. The problem is to determine the electric potential distribution inside a quadrant electrometer whose dimensions are shown in Fig. 7. The region of integration is bounded externally by a short closed circular cylinder divided axially into four quadrants, and internally by a plane vane, parallel to and midway between the ends of the cylindrical box. The shape of this vane is shown

† This derivation of the 'irregular star' residual formula differs from that given in two dimensions by Southwell (**2**), because one cannot, in three dimensions, make use of the analogy which interprets w as the transverse deflexion of a uniformly tensioned string-network.

‡ It should perhaps be mentioned that across a plane of symmetry (7) is an *exact* relation whereas across a true boundary it is only the usual *approximation* to the condition $\partial w/\partial \nu = 0$.

in the plan view (Fig. 7 a); its position along the axis of the box is shown in the elevations (Fig. 7 b and c).

The equation to be satisfied by the electric potential V, in the region between the vane and the box, is again Laplace's equation, $\nabla^2 V = 0$. The

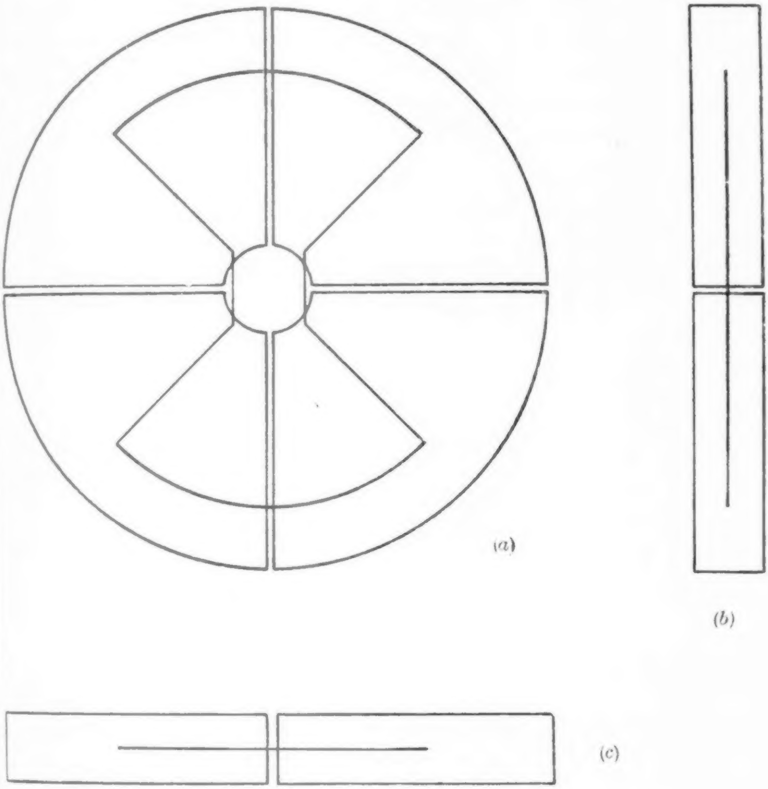

FIG. 7.

boundary conditions given are that the potential is maintained at prescribed values on the vane (V_0) and on each pair of opposite quadrants of the box (V_1 and V_2). In computation these values were taken to be

$$\left.\begin{aligned}&\text{on the vane, } V_0 = 200,\\&\text{on one pair of quadrants, } V_1 = 4,\\&\text{on the other pair of quadrants, } V_2 = -4.\end{aligned}\right\} \qquad (9)$$

Because the thickness of the box is small compared with the diameter of its cross-section, a smaller mesh-length ($h/4$) was used in the direction

parallel to its axis than the length (h) in the two perpendicular directions; this circumstance implies a modified residual-formula,

$$F_0 \equiv V_1+V_2+V_3+V_4+16V_5+16V_6-36V_0, \tag{10}$$

and a correspondingly modified relaxation-pattern. On the finest lattice used, there were four of the smaller mesh-lengths in the half-thickness of the box and eight of the larger mesh-lengths in the radius. The results are shown in Fig. 8 which records values, at the lattice-joints, of the potential, V. Conditions of symmetry make it sufficient to give results over one-quarter only of the whole volume.

REFERENCES

1. C. J. Tranter, *Quart. J. Mech. and Applied Math.* **1** (1948), 281.
2. R. V. Southwell, *Relaxation Methods in Theoretical Physics* (Oxford, 1946).
3. H. S. Carslaw, *The Conduction of Heat in Solids* (Macmillan, 1921).

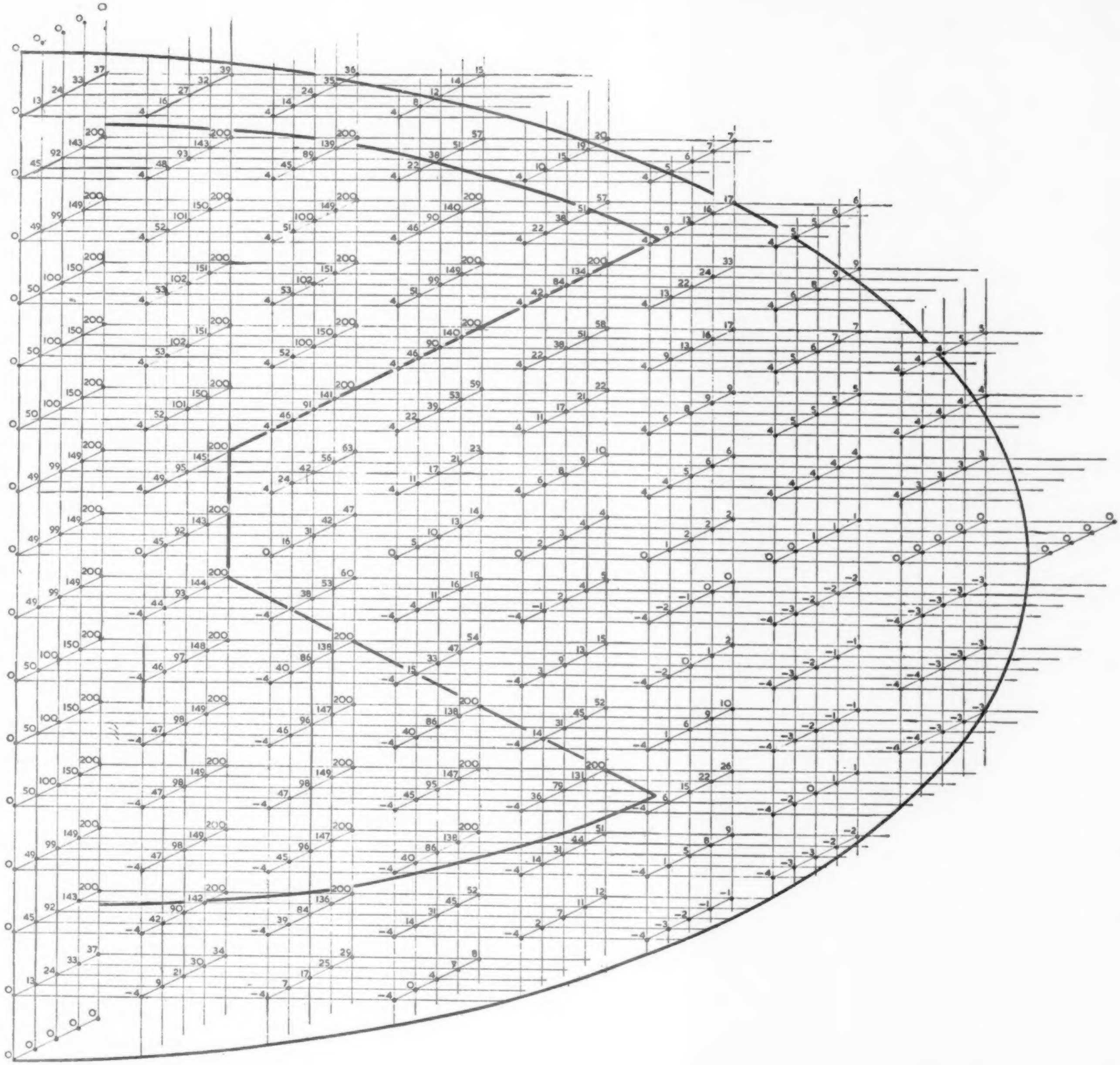

FIG. 8.

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THE APPLICATION OF RELAXATION METHODS TO THE SOLUTION OF NON-ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

I. THE HEAT-CONDUCTION EQUATION

By D. N. DE G. ALLEN *and* R. T. SEVERN

(*Imperial College, London*)

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SUMMARY

Relaxation methods have been applied in recent years to obtain solutions in two dimensions to various second-order linear partial differential equations. An example is provided by Laplace's equation† which, though simple, is typical of all the equations which have been solved in that it is of the so-called 'elliptic' form and in that all solutions which have been found have been required to satisfy a single boundary condition at all points of a closed boundary. There are many problems, however, for which the relaxation method has not hitherto been applicable because either one or (usually) both of these properties do not hold. Typical examples of these are provided when solutions are required of the heat-conduction equation (which is of 'parabolic' form) and of the wave equation (which is of 'hyperbolic' form); and also, occasionally, of Laplace's equation itself (which, although of 'elliptic' form, may not be associated with a single boundary condition at all points of a complete boundary).

In one dimension inherently the same difficulty distinguishes what are usually known as 'jury' problems from 'marching' problems. A method of overcoming these difficulties is presented here; the method was originally devised in order to bring the heat-conduction equation within the scope of relaxation and this paper is principally concerned with that application; the underlying idea is, however, introduced, for simplicity only, by illustrations in one dimension.

1. Relaxation methods were first devised, and for many years developed, as an alternative to classical analysis in order to find particular solutions of differential equations. By now, however, their procedures have been so far extended as sometimes to influence the general plan of attack. One example, illustrated by Fox and Southwell (**2**), in the determination of extensional stresses in a flat elastic plate, is the more convenient employment of two simultaneous differential equations instead of an orthodox reduction of them to a single equation of higher (i.e. biharmonic) order.

That tendency is maintained in this paper by the introduction, *solely on grounds of relaxational convenience*, of operations which would not be contemplated in classical analysis because they would make the problem not easier, but in fact harder, to solve. Thus, in order to make the (heat-conduction) equation,

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \tag{1}$$

† A very full discussion has been given by Southwell (**1**).

tractable by relaxation, a function w is introduced which is related to v by the equation

$$v = \frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2}; \tag{2}$$

elimination of v between equations (1) and (2) leads to a modified problem governed by the equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^4 w}{\partial x^4}, \tag{3}$$

with, of course, appropriate boundary conditions. Clearly, equation (3) is harder than equation (1) to solve by the methods of classical analysis. In contrast, the device just outlined makes this problem *for the first time* soluble by relaxation. An example is worked in detail in sections 9–10 with results which are presented in Fig. 1.

2. The difference between 'jury' and 'marching' problems was discussed by Richardson (**3**) who gave them those names. In one dimension a marching problem is one in which the end-conditions to be imposed are all stated at one end of the range of integration of the differential equation to be solved; and numerical solutions are usually constructed by means of a 'step-by-step' integration process. In a jury problem, on the other hand, the conditions are imposed in equal numbers at both ends; classical analysis then must have recourse to a synthesis of solutions which in numerical work are found, successively, by a marching process; as such they are liable to accumulated error, and the fact that the relaxation method enables jury problems to be solved *directly* is one of the features which gives it claim to importance. (The problem being, so to speak, 'tethered' at both ends, or—in two dimensions—all round a boundary, the solution cannot 'take charge' anywhere in the region.)

It is not the intention of this paper to suggest that 'step-by-step' methods of integration are not perfectly satisfactory to solve many kinds of marching problems, nor is it suggested that the method described here necessarily obtains more accurate results; only experience will in time indicate which method is the more suitable to employ for given types of problem. In particular the numerical examples given as illustrations in this paper certainly yield satisfactorily to a 'step-by-step' attack—indeed, for that matter, exact solutions can be found for them by analytical means; they have been chosen here as intrinsically simple illustrations of an alternative method which is being developed to solve much more complicated problems, some of which present great difficulties to an approach either by orthodox analysis or by 'step-by-step' operations.

3. It is, in fact, only to problems of jury type that the relaxation method is directly applicable; and the significance of the procedure outlined in

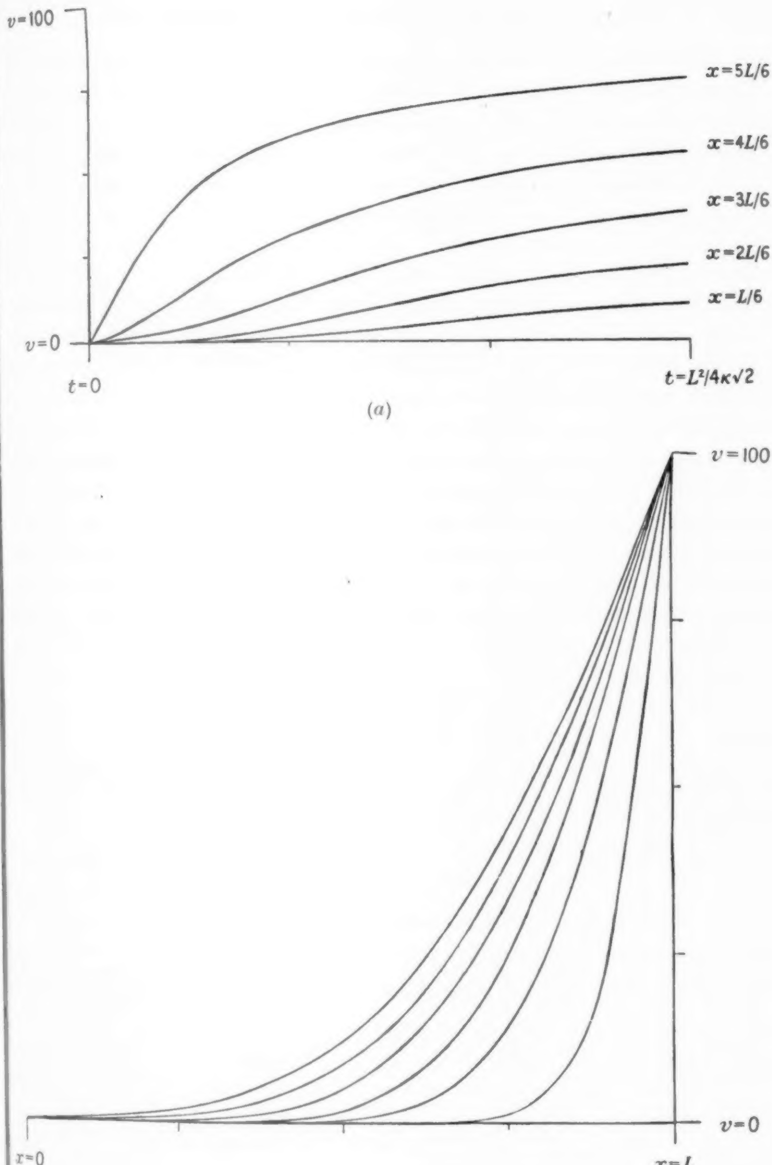

FIG. 1. The distribution of temperature (v) in a rod when one end is suddenly maintained at $v = 100$, the other end being kept at the initial temperature of the whole rod, $v = 0$. (a) The variation of temperature with time (t) at five equally-spaced points of subdivision along the rod; (b) the variation of temperature with position (x) at instants of time, $t = (L^2/144\kappa\sqrt{2})(1, 3, 5, 7, 9, 11)$.

section 1 is that it makes a jury problem of one which as presented was of marching type. It was devised in consequence of purely relaxational considerations of the circumstances which are necessary for solving a differential equation. Its effect would be nugatory in a classical attack, which would only yield, for instance, as a solution to equation (3):

$$w = \tfrac{1}{2}\int\left\{\int V\,dx\right\}dx+W, \tag{4}$$

where V denotes the general solution of equation (1), and W the general solution of equation (2) when v is put equal to zero. Then from (4), according to (2), it follows that $v = V$, simply, is the solution to equation (1) so that the introduction of the function w has achieved nothing. The utility of the procedure from a relaxational standpoint is only revealed when the differential equations are replaced by their finite-difference approximations.

4. For clarity it is well to consider firstly equations in one independent variable (x) only; a relaxational solution of such an equation consists of numerical values of the required quantity or dependent variable (v) at a number of points of subdivision equally spaced along the range of integration. In terms of v-values at such points, finite-difference approximations to low order v-derivatives are well known: Fig. 2 represents five typical

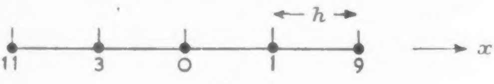

FIG. 2.

successive points of subdivision, and the approximations to the first four derivatives at the central point 0 are

$$\left.\begin{aligned} 2h(dv/dx)_0 &= v_1-v_3,\\ h^2(d^2v/dx^2)_0 &= v_1+v_3-2v_0,\\ 2h^3(d^3v/dx^3)_0 &= v_9-2v_1+2v_3-v_{11},\\ \text{and}\qquad h^4(d^4v/dx^4)_0 &= -4v_1-4v_3+6v_0+v_9+v_{11}, \end{aligned}\right\} \tag{5}$$

where suffixes denote values obtaining at correspondingly numbered points, and h is the length of the interval between successive points. It may be noticed that the approximations to the first and second derivatives contain values of v at points at distances up to, but not exceeding, one interval from 0; we shall say, briefly, that these approximations 'extend' over one interval on either side of 0. It is evident that, in general, the approximations to the derivatives of orders $2n-1$ and $2n$ both similarly extend over n intervals on either side of the central point.

If, then, the governing equation is of order either $2n-1$ or $2n$, its finite-difference approximation, holding at the point 0, extends over n intervals

on either side of 0; and thus when 0 represents either end-point of the range the finite-difference approximation involves n 'fictitious' v-values at n fictitious points outside the range. As an essential preliminary to the use of relaxation, these fictitious values have to be eliminated; thus n conditions are required at each end that this elimination may be achieved. In a jury problem this requirement is satisfied and a relaxational solution can be directly effected. But in a marching problem, *as presented*, no conditions are available at one end—we have instead some unwanted conditions (additional to the number that are wanted) at the other end; a relaxational technique cannot then be directly applied.

Since we have just seen that relaxation methods can only be utilized to solve a problem when it is expressed in jury form, and that then the order of the equation is double the number of conditions available at either end of the range, it follows conversely that if m end-conditions are *required to be imposed* at one end, then the governing equation—if not of order $2m$ as presented—must be transformed into a modified equation of that order before a relaxational technique can be used. In general m conditions may be given at one end and p conditions at the other, and it is sufficient to suppose that $p \leqslant m$; the order of the governing equation as presented would then be $m+p$. If $p = m$ the problem is of jury type and needs no further discussion. If $p < m$ the governing equation must be transformed so that its order is increased by $m-p$ to become $2m$; then it will be possible to impose m conditions at the one end while at the other end the original p conditions must be augmented by $m-p$ extra conditions and the problem will thus be converted into jury type. As an extreme case, in a marching problem m conditions are given at one end but none at the other; the transformation must now *double* the order of the governing equation from m to $2m$.

5. The transformation of the governing equation may be accomplished by means of a change of the dependent variable by substitution in the equation, in place of v, a differential function of a new dependent variable, w. We make no attempt to lay down rules according to which this substitution should be chosen. In any particular case there would be several possibilities which would suffice and a compromise must be made, firstly to make the substitution itself as simple as possible,† and secondly to make the transformed equation as convenient as possible for solution by relaxation. It is apparent that the order of the differential function of w which must be substituted for v is equal to the desired increase in order (i.e. to $m-p$) of the transformed equation above the original equation; and hence

† Thus keeping to a minimum the work which has to be done after the transformed equation has been solved.

is also equal to the number of extra end-conditions which have to be supplied. These extra end-conditions can be arbitrarily chosen provided only that they must not imply any restriction on v or v-derivatives; normally they would consist of assigned values to w and the first $(m-p-1)$ w-derivatives (such conditions can manifestly never be interpreted in terms of v).

The transformation of the governing equation could be accomplished by an alternative method which does not require the substitution for v of a differential function of a new variable, w, but instead requires us to operate through that equation with a differential operator so as to increase its order by the desired amount. This differential operator has to be chosen with aims similar to those which govern the choice of the substitution in the first method, and in the worked examples of this paper there would be little to choose between the two methods. We shall, however, adopt the first method here, partly because it is perhaps the less obvious of the two and therefore the more desirable to emphasize; and partly because it has been found to be the more convenient method to use for more advanced problems.

6. As an example to illustrate the procedure we consider the equation

$$d^2v/dx^2 = f(x), \tag{6}$$

to be solved over a range of integration $a \leqslant x \leqslant b$; $f(x)$ is a function of x which is supposed known only in the sense that its numerical value can be calculated or measured for any value of x within that range. In order to define a particular solution of (6) two end-conditions must be given; if these are stated one at each end of the range, an immediate and straightforward application of the relaxation method will yield the solution to any required degree of accuracy. But if the conditions are both given at one end of the range, as for instance, that at

$$\left.\begin{array}{ll} x = a, & v = k \\ \text{and} & dv/dx = K, \end{array}\right\} \tag{7}$$

then the problem, as presented, is of marching type and the equation must be transformed into another of double the original order before relaxation can be applied.

A simple and sufficient substitution is given by the relaxation

$$v = d^2w/dx^2, \tag{8}$$

which transforms (6) into the modified equation

$$d^4w/dx^4 = f(x). \tag{9}$$

The given end-conditions (7) become, in terms of w, that at

$$\left.\begin{array}{ll} x = a, & d^2w/dx^2 = k \\ \text{and} & d^3w/dx^3 = K, \end{array}\right\} \tag{10}$$

while the two extra conditions to be supplied may be chosen, as simply as possible, to be that at

$$\left.\begin{array}{ll} x = b, & w = 0 \\ \text{and} & dw/dx = 0. \end{array}\right\} \tag{11}$$

The problem is now converted to be of jury type, requiring only the solution for w of equation (9) subject to the end-conditions (10) and (11); it can therefore be solved by relaxation. And when w has been so found, v can be at once derived by use of equation (8). It should be mentioned (cf. section 3) that any other choice for the extra conditions (11) would only alter the w-values by a quantity $A+Bx$ (where A and B are some constants) and would therefore not make any difference in v.

If the alternative (second) method of transforming the governing equation were used for this example, it would require (6) to be replaced by the modified equation

$$d^4v/dx^4 = f''(x);$$

the end-conditions (7) would be unaltered, and the extra end-conditions at $x = b$ would comprise (6) itself, together with

$$d^3v/dx^3 = f'(x).$$

In this method the extra end-conditions are, of course, no longer arbitrary, but must instead be derived from the original governing equation.

7. As a second example in one dimension we take for solution the first-order equation

$$\frac{dv}{dx} = x+v, \tag{12}$$

over the range $0 \leqslant x \leqslant 1$ and with the end-condition given to be that at

$$x = 0, \qquad v = 1. \tag{13}$$

Again the order of the equation must be doubled by a preliminary transformation. In this case an appropriate substitution is

$$v = dw/dx+w; \tag{14}$$

thus (12) is transformed to become

$$d^2w/dx^2-w-x = 0, \tag{15}$$

and, in terms of w, the end-condition (13) becomes that at

$$x = 0, \qquad dw/dx+w = 1. \tag{16}$$

The extra condition is chosen to be simply that at

$$x = 1, \qquad w = 0. \tag{17}$$

The problem is now converted into jury type—to solve the equation (15) for w subject to the conditions (16) and (17). The solution for w (multiplied by 1,000 to avoid decimals) is shown in Fig. 3; it was easily effected by the normal relaxation procedure in less than 30 minutes, with the range divided into ten intervals. Fig. 3 also records (in brackets) values

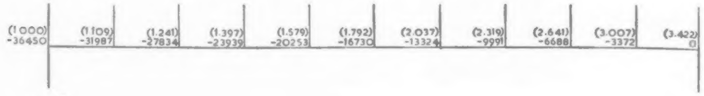

FIG. 3.

of v computed afterwards from the finite-difference approximation to (14),

$$v_0 = 5(w_1 - w_3) + w_0.$$

These values of v have been compared with the exact values (derived from the analytical solution $v = 2e^x - x - 1$); they are nowhere in error by as much as $\frac{1}{2}$ per cent. They are also negligibly different from the values which are obtained when the alternative (second) method of transforming the governing equation, which is suggested in section 5, is employed.

8. Having explained and illustrated the methods of this paper in one dimension we now proceed to apply them in two dimensions to the solution of the heat-conduction equation

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}, \tag{18}$$

where v, a function of time, t, and distance, x, measures the temperature in a linear flow of heat; κ is the diffusivity of the medium.

Two difficulties present themselves if an attempt is made to solve this equation directly by relaxation. A rectangular net would be used to cover the region of integration, and Fig. 4 shows a typical group of seven nodal points on such a net; the mesh-lengths in the x- and t-directions are denoted respectively by h and H. The finite-difference replacement of equation (18), holding at the typical node, 0, can be written down by use of the approximations (5), and leads to the definition of a residual there by the expression

$$F_0 \equiv v_1 + v_3 - 2v_0 - (h^2/2\kappa H)(v_2 - v_4). \tag{19}$$

If we conveniently choose mesh-lengths so that $h^2 = 2\kappa H$ (as we clearly may, cf. section 9), then the residual formula (19) yields a corresponding relaxation-pattern (illustrated in Fig. 5), which shows the changes made to the residuals when a unit increment is added to the value of v at a typical node of the net.

The first difficulty is apparent in this relaxation pattern; residuals are easily moved about in the x-direction, but it may be observed that an increment added to a v-value at any node makes no change at all to the *total* of the residuals on the mesh-line through that node in the x-direction.

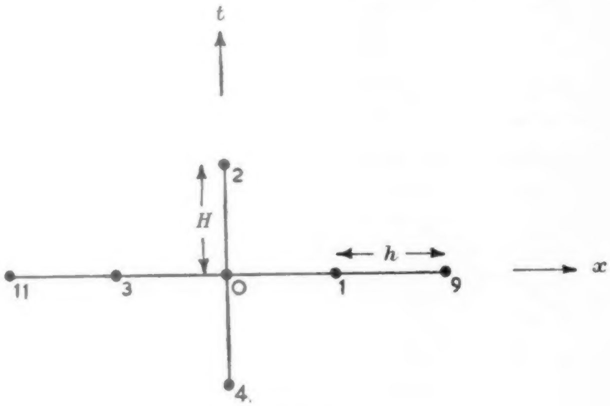

FIG. 4.

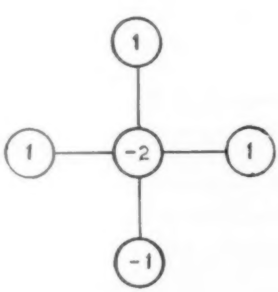

FIG. 5.

Thus the pattern does not easily allow residuals to be moved from one node to another in the t-direction. This difficulty would not, in itself, be unsurmountable; we take notice of it here only because we shall have later an (incidental) opportunity of removing it.

The second difficulty cannot be overcome, however, except by a preliminary transformation of the governing equation (18); it is that the equation is of marching type in the time coordinate. The principal object of the transformation is therefore to double the order of (18) in the t-derivatives, but at the same time it is chosen so far as possible to make the transformed equation both simple and convenient for the application

of relaxation; the substitution

$$v = \frac{\partial w}{\partial t} + \kappa \frac{\partial^2 w}{\partial x^2}, \tag{20}$$

is an obvious choice. It transforms (18) into the form

$$\frac{\partial^2 w}{\partial t^2} - \kappa^2 \frac{\partial^4 w}{\partial x^4} = 0, \tag{21}$$

and the finite-difference approximation to this equation, on a rectangular net of sides h and H in the x- and t-directions, is

$$(h^4/\kappa^2 H^2)(w_2 + w_4 - 2w_0) + 4w_1 + 4w_3 - 6w_0 - w_9 - w_{11} = 0. \tag{22}$$

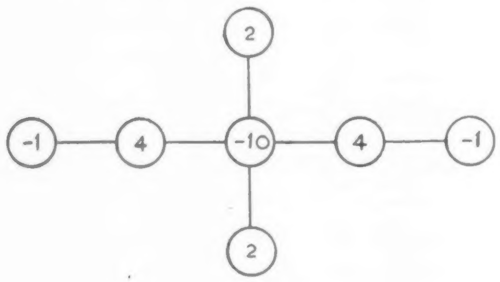

FIG. 6.

The relaxation pattern found from this approximation does not suffer from the disadvantage which was apparent in that of Fig. 5; we can indeed by a suitable choice of the mesh-interval, H, make the pattern derived from equation (22) such that the effect of an increment added to w at a node of the net is to move the residual away from that node equally in all four directions. To do this requires that we choose

$$H = h^2/\kappa\sqrt{2}, \tag{23}$$

and then equation (22) leads to residuals being defined by the expression

$$F_0 \equiv 2w_2 + 2w_4 + 4w_1 + 4w_3 - 10w_0 - w_9 - w_{11}. \tag{24}$$

The corresponding relaxation pattern is shown in Fig. 6.

9. We take as an illustrative example the determination of the temperature, in a medium which extends from $x = 0$ to $x = L$, which is initially at zero temperature throughout; where the end $x = 0$ is kept at zero temperature, but where at time $t = 0$ the other end $x = L$ is raised to and subsequently maintained at a temperature of 100. In terms of w this problem requires the solution of the equation (21) over a rectangular region in the (x, t)-plane bounded by the lines $x = 0$, $x = L$, $t = 0$, and $t = T$;

T measures the extent in time for which the solution is required. The boundary conditions to be imposed are that

$$\left.\begin{array}{lll} & \text{on } x=0, & \dfrac{\partial w}{\partial t}+\kappa\dfrac{\partial^2 w}{\partial x^2}=0, \\ & \text{on } x=L, & \dfrac{\partial w}{\partial t}+\kappa\dfrac{\partial^2 w}{\partial x^2}=100, \\ \text{and} & \text{on } t=0, & \dfrac{\partial w}{\partial t}+\kappa\dfrac{\partial^2 w}{\partial x^2}=0. \end{array}\right\} \tag{25}$$

When a solution for w has been found satisfying equation (21) and the conditions (25), then v, when calculated from the relation (20), must automatically satisfy all requirements. But to find a solution for w we need some extra boundary-conditions—one on each of the lines $x = 0$, $x = L$, and $t = T$. These are arbitrarily chosen to be:

$$\left.\begin{array}{lll} & \text{on } x=0, & w=0, \\ & \text{on } x=L, & w=0, \\ \text{and} & \text{on } t=T, & \partial w/\partial t=0. \end{array}\right\} \tag{26}$$

In choosing these extra conditions we have only to ensure that they cannot be interpreted as implying any condition on v or its derivatives, and that they are everywhere consistent with one another and with the other conditions (25). It follows, incidentally, that, by virtue of their being so chosen, the conditions (25) therefore simplify to become:

$$\left.\begin{array}{lll} & \text{on } x=0, & \partial^2 w/\partial x^2=0, \\ & \text{on } x=L, & \kappa\,\partial^2 w/\partial x^2=100, \\ \text{and} & \text{on } t=0, & \partial w/\partial t+\kappa\partial^2 w/\partial x^2=0. \end{array}\right\} \tag{27}$$

10. In finite-difference terms the second of the conditions (27) becomes

$$(\kappa/h^2)(w_1+w_3-2w_0) = 100;$$

so that it is convenient to work numerically not in terms of w but in terms of a multiple $\kappa w/h^2$ ($= w'$, say). Then the relation (24) is replaced by

$$F_0 \equiv 2w_2'+2w_4'+4w_1'+4w_3'-10w_0'-w_9'-w_{11}', \tag{28}$$

and the finite-difference replacements for the boundary conditions (26) and (27) are that

$$\left.\begin{array}{llll} & \text{on } x=0, & w_0'=0, \text{ and } & w_1'+w_3'=0, \\ & \text{on } x=L, & w_0'=0, \text{ and } & w_1'+w_3'=100, \\ & \text{on } t=0, & \multicolumn{2}{l}{w_2'-w_4'+(w_1'+w_3'-2w_0')\surd 2=0,} \\ \text{and} & \text{on } t=T, & w_2'=w_4'. & \end{array}\right\} \tag{29}$$

These conditions are just sufficient for the elimination of all the fictitious values which occur in the use of (28) in order to calculate the residuals. We can thus find w' as a purely numerical quantity and, when it has been evaluated, v can also be found, without specifying any particular values

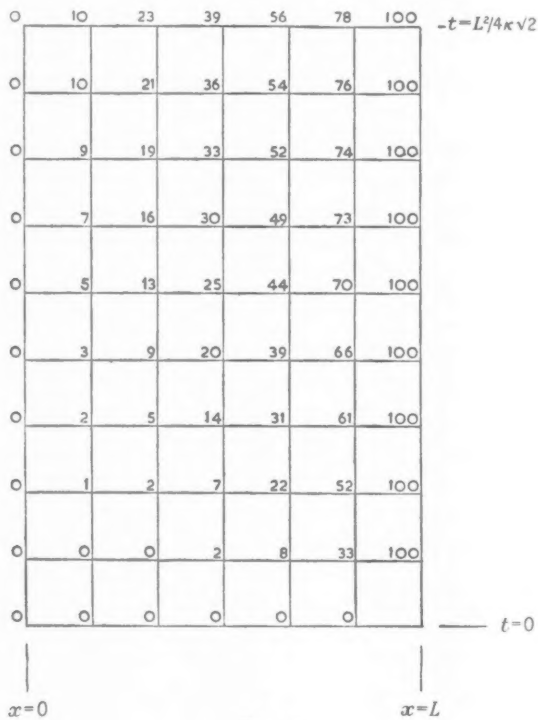

FIG. 7.

for L and κ; for, replacing (20) by its finite-difference approximation, using (23) and substituting for w in terms of w', we find that

$$v_0 = w_1'+w_3'-2w_0'+(w_2'-w_4')/\sqrt{2}. \tag{30}$$

The solution is, of course, not independent of the values of L and κ; but these values only fix the time-interval of the rectangular net at the nodes of which v is determined. This interval is given by equation (23); when n mesh-lengths are used in the x-direction it follows from that equation that the mesh-length in the time-direction is $L^2/n^2\kappa\sqrt{2}$, and therefore that, if N mesh-lengths are used in this direction, the total time-range covered by the solution is $T = NL^2/n^2\kappa\sqrt{2}$.

The solution has been obtained on two nets of different sizes. In the first, or coarser, net the space-range, L, was divided into six mesh-lengths ($n = 6$) and the time-range extended over nine mesh-lengths ($N = 9$). Thus the total time-range was $L^2/4\kappa\sqrt{2}$. It has not been thought necessary to record values of w' and Fig. 7 presents simply the nodal-values of v

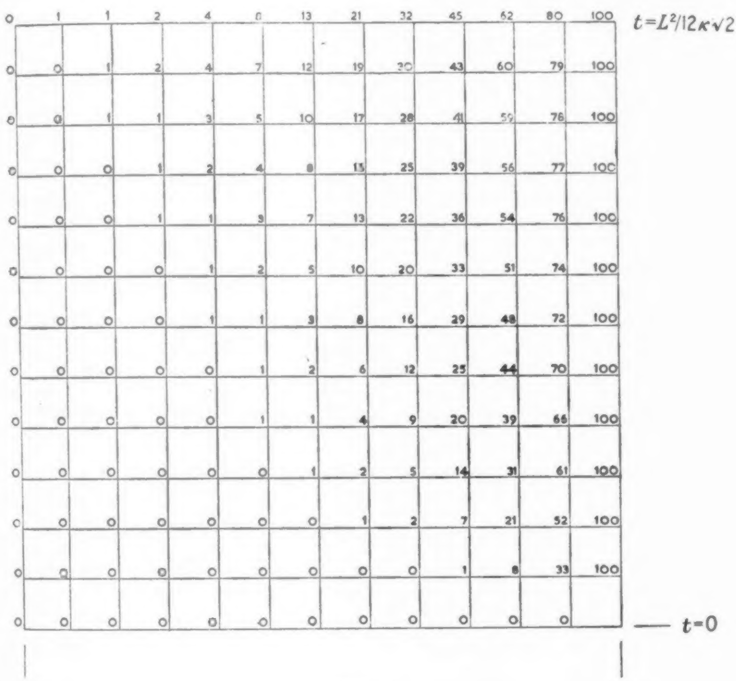

FIG. 8.

deduced by use of equation (30). These results are presented graphically in Fig. 1 in which the curves show, for different values of x, the variation of temperature with time.

On the finer net twelve mesh-lengths were taken in the x-direction and also in the t-direction ($n = N = 12$). In this case the total time-range was only $L^2/12\kappa\sqrt{2}$, but within that range much greater detail of the solution is obtained. The results are recorded in Fig. 8. From these values curves showing the variation of temperature with x, for different values of time, were drawn and are also shown in Fig. 1.

This particular example of a solution of the heat-conduction equation can, of course, be worked analytically by a use of Fourier series; it has been

chosen here only to illustrate the methods of this paper. The exact solution allows, however, of a check of the accuracy of the numerical method: the smoothness of the curves obtained and shown in Fig. 1 indicates that it is unnecessary to check all of the values given in Figs. 7 and 8; and in a number that have been checked the errors do not exceed 3 per cent. of the maximum temperature.

REFERENCES

1. R. V. Southwell, *Relaxation Methods in Theoretical Physics* (Oxford, 1946).
2. L. Fox and R. V. Southwell, *Phil. Trans. R.S.* C, **1** (1941), 15.
3. L. F. Richardson, ibid. A, **226** (1927), 299.

TABLES OF TWO INTEGRALS AND OF SPIELREIN'S INDUCTANCE FUNCTION

By ALAN FLETCHER

(*Department of Applied Mathematics, The University, Liverpool*)

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SUMMARY

Tables are given for $\alpha = 0(\cdot 01)1$ of the integrals

$$\int_\alpha^1 (K-E)\,dk \quad \text{and} \quad \int_\alpha^1 (K-E)k^{-3}\,dk,$$

where K and E are the complete elliptic integrals with modulus k, and also of an expression due to Spielrein which involves both of the above integrals and occurs in connexion with the self-inductance of thin disk coils. Tables are also given of certain auxiliary functions and of coefficients required in their calculation.

1. Account of previous work

THIS paper is mainly concerned with the use and extension of formulae of known type; no adequate connected account being available, a concise summary is necessary.

The functions to be tabulated arise in connexion with the self-inductance of thin disk coils. If the windings consist of a very large number, N, of almost circular turns evenly spaced in one plane between two concentric limiting circles of radii a and A, the self-inductance in absolute electromagnetic units may be written as $L = N^2 A f(\alpha)$, where $f(\alpha)$ is a function of the single parameter $\alpha = a/A < 1$. A 'current-sheet' assumption of the usual kind is made; correction for thickness of wire, finite spacing between turns, and so on, is possible but is neglected here. With such assumptions, $f(\alpha)$ is a definite function, but its mathematical determination encountered difficulties, and only approximations were at first available. An exact expression was, however, published in 1915 by Spielrein (**1**) and also in 1919 by Butterworth (**2**); this expression, which will be called Spielrein's inductance function, may be written, introducing the notations I and J for convenience,

$$f(\alpha) = \frac{16\pi}{3(1-\alpha)^2}(I-\alpha^3 J), \tag{1}$$

where
$$I = \int_\alpha^1 (K-E)\,dk, \qquad J = \int_\alpha^1 (K-E)k^{-3}\,dk, \tag{2}$$

K and E being the complete elliptic integrals of the first and second kinds with modulus k. From his exact formula Spielrein deduced series approximations both for small α (say $\alpha < 0{\cdot}5$) and for large α (say $\alpha > 0{\cdot}5$). He also constructed some useful but sketchy tables of $f(\alpha)$.

For the larger values of α an excellent approximation, found by a different method, had already been published in 1914 by Lyle (**3**), extending an earlier approximation due to Rayleigh and Niven; see (**4**) and (**5**). Correcting a known error (p. 435, for 4298579 read 98579), Lyle's expression, if rewritten as a formula for $f(\alpha)$, with Lyle's c/a replaced by 2γ, becomes

$$f(\alpha) = 2\pi(1+\alpha)\left(S_1 \ln\frac{4}{\gamma} - S_2\right), \tag{3}$$

where

$$\left.\begin{aligned} S_1 &= 1+\frac{1}{24}\gamma^2+\frac{11}{2880}\gamma^4+\frac{103}{2^{10}.3.5.7}\gamma^6, \\ S_2 &= \frac{1}{2}-\frac{43}{288}\gamma^2-\frac{1}{150}\gamma^4-\frac{98579}{2^{13}.3^2.5^2.7^2}\gamma^6, \end{aligned}\right\} \tag{4}$$

and

$$\gamma = \frac{1-\alpha}{1+\alpha} = \frac{\text{radial width}}{\text{mean diameter}}. \tag{5}$$

This appears to be the best previously published approximation for large α. It should be added that Lyle's paper was of wider electrical scope than Spielrein's; he investigated coils of mean radius a and rectangular section, the axial dimension of the rectangle being b and the radial dimension c. Thus formulae (3, 4, 5) relate merely to a limiting case, $b = 0$, in his work.

Formulae similar to (3, 4, 5) were deduced by Spielrein from his exact expression by an application of Landen's transformation. He carried the approximation to a lower order than Lyle, in that he did not obtain the terms in γ^6 in (4). On the other hand, it was Spielrein's exact expression which enabled the limiting case of a disk coil to be seen in its proper mathematical perspective, and his work is basic for the present paper, which adds terms in γ^8 and γ^{10}.

Some details of the effect of the Landen-type transformation (5) will be required. In the I, J notation used here, Spielrein's results, which are easily checked by the usual Landen formulae for complete integrals (**6**), show that

$$I = \int_0^\gamma \frac{(1+\gamma^2)K-2E}{(1+\gamma)^3}\,d\gamma, \qquad J = \int_0^\gamma \frac{(1+\gamma^2)K-2E}{(1-\gamma)^3}\,d\gamma, \tag{6}$$

where K and E now have modulus $\sqrt{(1-\gamma^2)}$. On using the usual expressions (**7**) of the form

$$K = K_1 \ln(4/\gamma) - K_2, \qquad E = E_1 \ln(4/\gamma) + E_2, \tag{7}$$

where K_1, K_2, E_1, E_2 are power series in γ^2, and performing on series the various operations demanded by (6), one obtains formulae, essentially due to Spielrein, which will be written in the form

$$I = I_1 \ln(4/\gamma) - I_2, \qquad J = J_1 \ln(4/\gamma) - J_2, \tag{8}$$

where

$$\left.\begin{aligned} I_1 &= \gamma-\frac{3}{2}\gamma^2+\frac{25}{12}\gamma^3-\frac{43}{16}\gamma^4+\frac{1057}{320}\gamma^5-\frac{1507}{384}\gamma^6+\dots, \\ I_2 &= \gamma-\frac{9}{4}\gamma^2+\frac{29}{9}\gamma^3-\frac{265}{64}\gamma^4+\frac{16131}{3200}\gamma^5-\frac{6835}{1152}\gamma^6+\dots, \end{aligned}\right\} \tag{9}$$

and J_1, J_2 have the same terms with all signs positive, i.e.

$$J_1(\gamma) = -I_1(-\gamma), \qquad J_2(\gamma) = -I_2(-\gamma). \tag{10}$$

Terms involving γ^7 to γ^{12} are added to (9) below.

Using (5), equation (1) may be written

$$f(\alpha) = 2\pi(1+\alpha)\frac{(1+\gamma)^3I-(1-\gamma)^3J}{3\gamma^2}. \tag{11}$$

Substituting (8, 9, 10) in (11), one finds, with Spielrein, Lyle's formulae (3, 4) with the γ^6 terms of S_1 and S_2 omitted.

Integrals of complete elliptic integrals

In describing previous work, it remains only to consider the type of integral occurring in (2). The integrals I and J could be replaced by others, using relations mentioned by Spielrein and examined in detail by Müller (8); see also Jahnke and Emde (9). In virtue of the known differential relations

$$\frac{dK}{dk} = \frac{E}{k(1-k^2)}-\frac{K}{k}, \qquad \frac{dE}{dk} = \frac{E-K}{k}, \tag{12}$$

it is shown by Müller that recurrence formulae exist which, in general, connect $\int Kk^r\,dk$ with $\int Kk^{r-2}\,dk$, and $\int Ek^r\,dk$ with $\int Ek^{r-2}\,dk$; but these recurrence formulae give no connexion (owing to vanishing of coefficients) between $\int K\,dk$ and $\int Kk^{-2}\,dk$, nor between $\int Kk\,dk$ and $\int Kk^{-1}\,dk$, and similarly when K is replaced by E. Moreover,

$$2\int E\,dk = \int K\,dk+kE, \qquad \int Ek^{-1}\,dk = \int Kk^{-1}\,dk+E.$$

Thus, restricting ourselves to integral indices, the results are that $\int Kk^r\,dk$ and $\int Ek^r\,dk$ may be expressed rationally in terms of:

(1) k, K, E if $r = 1, 3, 5,\dots$ or $-2, -4, -6,\dots,$

(2) $k, K, E, \int K\,dk$ if $r = 0, 2, 4,\dots,$

(3) $k, K, E, \int Kk^{-1}\,dk$ if $r = -1, -3, -5,\dots$.

Hence all integrals of the form $\int Kk^r\,dk$ and $\int Ek^r\,dk$, where r is integral, may be evaluated in terms of, say, $\int K\,dk$ and $\int Kk^{-1}\,dk$, together with k, K, E.

We also have $$\int_0^1 K\,dk = 2G, \qquad \int_0^1 E\,dk = G+\tfrac{1}{2}, \tag{13}$$

where G is Catalan's constant, defined by

$$G = \frac{1}{2}\int_0^{\frac{1}{2}\pi} \frac{\phi\,d\phi}{\sin\phi} = \int_0^1 \tan^{-1}t\,\frac{dt}{t}$$
$$= 1-3^{-2}+5^{-2}-7^{-2}+\ldots = 0{\cdot}91596\ 55941\ 77219 \tag{14}$$

It follows from (2) that

$$I(0) = \int_0^1 (K-E)\,dk = G-\tfrac{1}{2}, \tag{15}$$

and from (1) that

$$f(0) = \tfrac{8}{3}\pi(2G-1) = 6{\cdot}96957\ 04257. \tag{16}$$

In considering the whole class of integrals of the above kind, it might be simplest to consider $\int K\,dk$ and $\int Kk^{-1}\,dk$ as the basic integrals in terms of which the others may be expressed. But the pair I and J which occur directly in Spielrein's formula have themselves an elegant mutuality visible in equations (10), and will be tabulated below. In terms of them, we have

$$\left.\begin{aligned}
\int_\alpha^1 K\,dk &= 2I-\alpha E+1, \\
\int_\alpha^1 E\,dk &= I-\alpha E+1, \\
\int_\alpha^1 Kk^{-1}\,dk &= 2J-(1-\alpha^2)\alpha^{-2}K+\alpha^{-2}E-1, \\
\int_\alpha^1 Ek^{-1}\,dk &= 2J-(1-\alpha^2)\alpha^{-2}(K-E),
\end{aligned}\right\} \tag{17}$$

where the complete integrals K and E on the right have modulus α. Thus it would be easy to tabulate other integrals of the class if desired.

2. Remarks on previous work

The excellent pioneer analysis of Spielrein was accompanied by two less attractive numerical tables of $f(\alpha)$. Firstly, he gave $f(\alpha)$ to 4–6 decimals for $\alpha = 0(\cdot05)\cdot9$. Only 4 decimals are given in the middle of the range, where values are hardest to calculate. The sixth decimal, when given, is always wrong by from 3 to 8 units, and is thus almost worthless; even the value of $f(0)$ is wrong by 3 units of the sixth place, although Spielrein gives G to 11 decimals and thus has material to determine $f(0)$, by (16), to about 10 decimals. Secondly, he gave $f(\alpha)$ to 1–3 decimals for $\alpha = \cdot05(\cdot01)\cdot95, \cdot99$.

The first decimal is wrong at $\alpha = \cdot 99$. It will be noticed that arguments are omitted, presumably as unimportant in practice, near 0 and 1. The erratic number of decimals given makes interpolation troublesome.

The case of a thin disk coil is admittedly less important in electrical practice than the well-known and mathematically easier case of a thin cylindrical coil. Nevertheless, it has its place; thus it is noticeable that Lyle (**3**), wishing to check his formulae for coils of rectangular section, was able to do so only at the cylindrical limit ($c = 0$), and not at the disk limit ($b = 0$), because no expressions for disk coils better than his own were then available. (The present paper will confirm that Lyle's coefficients as amended in (4) are correct as far as they go.) Spielrein's table only partially fills the gap. Grover in his recent book (**10**) has tabulated

$$P = 2f(\alpha)/(1+\alpha)$$

with argument γ; this variant form has considerable merits, and the table is useful when 5 figures suffice. For details of this and a few small tables, see Fletcher (**11**). A recomputation of $f(\alpha)$ seems not out of place.

There is a further, and very strong, reason for recomputation. As $f(\alpha)$ depends on *two* integrals, it is impossible to convert any table relating to disk coils so as to obtain numerical information about either integral separately. It seems a pity to refrain from tabulating two expressions which would bring a whole new set of integrals into the class of functions which may be regarded as known for computational purposes. The integrals I and J, or some other basic pair, should be tabulated, as well as an inductance function.

3. Aim of present paper

The present paper aims at tabulating I, J, $f(\alpha)$ for $\alpha = 0(\cdot 01)1$. A reason additional to those given above stimulated the author to undertake the necessary labour. In 1940 he constructed (**6**) a 10-decimal table of K and E for $k = 0(\cdot 01)1$; this arose out of tabulation of Laplace coefficients in dynamical astronomy, and at the time the author was unaware of any possible electrical application. This table appeared on reflection to offer a means of recalculating Spielrein's values by a totally different method, namely that of numerical quadrature of the integrals occurring in (2). The author was in the best position to carry this out, since he possessed manuscript 12-decimal working values of K and E for $k = 0(\cdot 01)\cdot 7(\cdot 005)1$, which would give a more accurate basis for numerical quadrature than the published 10-decimal values for $k = 0(\cdot 01)1$.

It was decided to tabulate I and J correct to 10 decimals, both for the sake of uniformity with the table (**6**) of K and E, and for the convenience of anyone who might wish to tabulate related integrals from equations such

as (17). In the case of $f(\alpha)$ there seemed no point in going beyond 6 decimals; this is the maximum number of decimals given (inaccurately) by Spielrein, and involves giving 7–8 significant figures, which are adequate for all electrical purposes. All quantities were to be calculated and, where possible, differenced to about three more figures than were to be printed.

It was found necessary to develop series for computation near singularities at $\alpha = 0$ and $\alpha = 1$, but numerical quadrature remained fundamental. In fact, I and J are otherwise not easily calculated in part of the range; the values given for them in the tables between $\alpha = \cdot 5$ and $\alpha = \cdot 8$ rest entirely on quadrature, since no formulae have been evolved which enable isolated values of I and J to be computed to about 12 decimals much inside this interval, although with further labour such formulae could be worked out.

Known series for small α are easily used to any number of terms. But the series for the larger values of α ($1-\alpha$ small) can be extended only with considerable labour. It was decided to give information regarding auxiliary functions in Tables 2 and 3, which relate to the case $1-\alpha$ small, except that the last third of Table 3 relates to the case α small.

4. Construction and description of tables

A. Low and medium values of α

The tables of I and J rest mainly on two highly accurate numerical quadratures carried out by Miss Olive E. Vanes.

From the author's manuscript values of K and E, 12-decimal values of $X = K-E$ were formed for $k = 0(\cdot 01)\cdot 7(\cdot 005)\cdot 93$, and differenced. The value of I at $\alpha = 0$ being $G-\frac{1}{2}$, numerical integration (using central even differences, up to the twelfth when necessary) gave $I = \int_\alpha^1 X\,dk$ for $\alpha = 0(\cdot 01)\cdot 75(\cdot 005)\cdot 90$, 14 decimals being retained. Checks were made at $\alpha = \cdot 1(\cdot 1)\cdot 5$ by the power series in α, the term in α^{35} being included at $\alpha = \cdot 5$, and at $\alpha = \cdot 88$, $\cdot 89$, $\cdot 90$ by the extension of the series (9) in γ described below. Agreement was found in all cases within 5 units of the fourteenth place. The 14-decimal values of I were checked by differencing. Table 1 gives I to 10 decimals at interval $\cdot 01$; differences are not printed, but are such that the table is interpolable by ordinary methods up to about $\alpha = \cdot 90$, naturally with high-order differences near this value of α.

In the case of J, a singularity occurs at $\alpha = 0$, because the integrand $Y = (K-E)k^{-3}$ has an infinite part $\frac{1}{4}\pi k^{-1}$ at $k = 0$. Thus $J+\frac{1}{4}\pi \ln \alpha$ was calculated to 13 decimals for $\alpha = 0(\cdot 01)\cdot 35$ from the power series in α, and checked by differencing; the results are given to 10 decimals at the end of Table 3, the corresponding values of J being inserted in Table 1.

Table 1

α	I	J	$f(\alpha)$	δ_m^2
0·00	0·41596 55942	∞	6·969570	
·01	·41596 53324	3·89700 58488	7·111010	
·02	·41596 34996	3·35256 51407	7·256441	3881
·03	·41595 85242	3·03403 99284	7·405760	3814
·04	·41594 88330	2·80799 17930	7·558896	3763
0·05	0·41593 28508	2·63260 25516	7·715798	3727
·06	·41590 89996	2·48924 52379	7·876430	3700
·07	·41587 56978	2·36798 36262	8·040762	3681
·08	·41583 13593	2·26288 66391	8·208777	3668
·09	·41577 43933	2·17012 85731	8·380461	3662
0·10	0·41570 32031	2·08709 72264	8·555808	3660
·11	·41561 61854	2·01192 93825	8·734816	3664
·12	·41551 17296	1·94324 92715	8·917489	3671
·13	·41538 82172	1·88001 20778	9·103833	3682
·14	·41524 40206	1·82140 56082	9·293860	3698
0·15	0·41507 75026	1·76678 59671	9·487586	3716
·16	·41488 70156	1·71563 40336	9·685029	3738
·17	·41467 09004	1·66752 51570	9·886210	3764
·18	·41442 74857	1·62210 75264	10·091156	3792
·19	·41415 50868	1·57908 63968	10·299894	3824
0·20	0·41385 20051	1·53821 23660	10·512458	3859
·21	·41351 65269	1·49927 25143	10·728881	3897
·22	·41314 69224	1·46208 36068	10·949202	3939
·23	·41274 14447	1·42648 68088	11·173462	3983
·24	·41229 83287	1·39234 35261	11·401706	4031
0·25	0·41181 57901	1·35953 20945	11·633982	4083
·26	·41129 20243	1·32794 51194	11·870342	4137
·27	·41072 52047	1·29748 73179	12·110839	4195
·28	·41011 34822	1·26807 37523	12·355532	4257
·29	·40945 49833	1·23962 83746	12·604484	4323
0·30	0·40874 78091	1·21208 28172	12·857759	4392
·31	·40799 00336	1·18537 53822	13·115427	4466
·32	·40717 97024	1·15945 01904	13·377562	4544
·33	·40631 48310	1·13425 64618	13·644242	4626
·34	·40539 34034	1·10974 79028	13·915548	4713
0·35	0·40441 33699	1·08588 21828	14·191569	4804
·36	·40337 26457	1·06262 04832	14·472394	4901
·37	·40226 91087	1·03992 71092	14·758121	5003
·38	·40110 05977	1·01776 91523	15·048853	5111
·39	·39986 49100	0·99611 61971	15·344697	5225
0·40	0·39855 97992	0·97494 00640	15·645766	5345
·41	·39718 29729	·95421 45845	15·952181	5471
·42	·39573 20898	·93391 54028	16·264070	5605
·43	·39420 47573	·91401 98002	16·581564	5747
·44	·39259 85286	·89450 65396	16·904808	5896
0·45	0·39091 08991	0·87535 57275	17·233949	6055
·46	·38913 93037	·85654 86900	17·569147	6222
·47	·38728 11128	·83806 78625	17·910569	6400
·48	·38533 36285	·81989 66902	18·258392	6588
·49	·38329 40810	·80201 95383	18·612805	6787
0·50	0·38115 96235	0·78442 16113	18·974008	6999

TABLE 1 (*continued*)

α	I	J	$f(\alpha)$	δ_m^2	Γ^4
0·50	0·38115 96235	0·78442 16113	18·974008	6999	
·51	·37892 73280	·76708 88790	19·342212	7224	
·52	·37659 41800	·75000 80098	19·717643	7463	
·53	·37415 70728	·73316 63094	20·100540	7718	
·54	·37161 28020	·71655 16643	20·491159	7990	
0·55	0·36895 80587	0·70015 24905	20·889771	8280	
·56	·36618 94224	·68395 76850	21·296667	8590	
·57	·36330 33538	·66795 65816	21·712158	8922	
·58	·36029 61860	·65213 89088	22·136575	9278	
·59	·35716 41157	·63649 47510	22·570274	9659	
0·60	0·35390 31931	0·62101 45109	23·013638	10070	
·61	·35050 93111	·60568 88744	23·467078	10512	
·62	·34697 81931	·59050 87763	23·931036	10988	
·63	·34330 53800	·57546 53674	24·405990	11504	
·64	·33948 62154	·56054 99823	24·892455	12062	
0·65	0·33551 58290	0·54575 41077	25·390991	12668	
·66	·33138 91194	·53106 93502	25·902204	13327	
·67	·32710 07329	·51648 74045	26·426755	14046	
·68	·32264 50419	·50200 00208	26·965365	14832	
·69	·31801 61187	·48759 89705	27·518821	15694	
0·70	0·31320 77077	0·47327 60111	28·087986	16642	
·71	·30821 31924	·45902 28486	28·673811	17688	
·72	·30302 55595	·44483 10972	29·277344	18845	
·73	·29763 73562	·43069 22357	29·899746	20131	
·74	·29204 06429	·41659 75592	30·542307	21565	
0·75	0·28622 69376	0·40253 81259	31·206464	23172	
·76	·28018 71519	·38850 46967	31·893831	24980	
·77	·27391 15165	·37448 76667	32·606220	27023	
·78	·26738 94940	·36047 69863	33·345685	29347	
·79	·26060 96754	·34646 20694	34·114557	32004	
0·80	0·25355 96578	0·33243 16852	34·915508	35062	
·81	·24622 58974	·31837 38295	35·751610	38607	1
·82	·23859 35321	·30427 55694	36·626430	42749	1
·83	·23064 61644	·29012 28548	37·544134	47629	1
·84	·22236 55946	·27590 02845	38·509638	53437	1
0·85	0·21373 14862	0·26159 08146	39·528796	60425	2
·86	·20472 09437	·24717 53879	40·608659	68937	2
·87	·19530 79702	·23263 24554	41·757827	79454	3
·88	·18546 27576	·21793 73478	42·986944	92668	4
·89	·17515 07415	·20306 14327	44·309406	109572	5
0·90	0·16433 13127	0·18797 09576	45·742401	131717	8
·91	·15295 60117	·17262 54177	47·308513		
·92	·14096 59195	·15697 51772	49·038253		
·93	·12828 77353	·14095 78663	50·974290		
·94	·11482 75938	·12449 26518	53·178968		
0·95	0·10046 07011	0·10747 05505	55·748695		
·96	·08501 25014	·08973 66646	58·844582		
·97	·06822 03460	·07105 36874	62·767886		
·98	·04964 08537	·05101 28237	68·193174		
0·99	·02834 58556	·02873 82572	77·266993		
1·00	0·0	0·0	∞		

TABLE 2

Coefficients in Expansions of Auxiliary Functions

The argument in the first column indicates the power of γ

	K_1	K_2	E_1	E_2
0	1·0	0·0	0·0	1·0
2	0·25	0·25	0·5	−0·25
4	0·14062 5	0·16406 25	0·1875	−0·20312 5
6	0·09765 625	0·12044 27083	0·11718 75	−0·14062 5
8	0·07476 80664	0·09488 42367	0·08544 92187	−0·10691 32487
10	0·06056 21338	0·07820 20569	0·06729 12598	−0·08614 34937
12	0·05088 90152	0·06648 24963	0·05551 52893	−0·07210 57892
14	0·04387 87937	0·05780 63778	0·04725 40855	−0·06199 33844
16	0·03856 53460	0·05112 77646	0·04113 63691	−0·05436 48807
18	0·03439 93364	0·04582 95359	0·03642 28268	−0·04840 63621
20	0·03104 54011	0·04152 45530	0·03267 93696	−0·04362 40574
22	0·02828 72353	0·03795 78437	0·02963 42465	−0·03970 12167
24	0·02597 90755	0·03495 47162	0·02710 86005	−0·03642 53767

	J_1		J_2	
1	1	1·0	1·0	1
2	3/2	1·5	2·25	9/4
3	25/12	2·08333 33333	3·22222 22222	29/9
4	43/16	2·6875	4·14062 5	265/64
5	1057/320	3·30312 5	5·04093 75	16131/3200
6	1507/384	3·92447 91667	5·93315 97222	6835/1152
7		4·54966 51786	6·82149 50043	
8		5·17724 60937	7·70758 05664	
9		5·80659 31532	8·59240 28938	
10		6·43717 65137	9·47639 39412	
11		7·06872 69731	10·35987 75585	
12		7·70100 27568	11·24300 07511	

	S_1		S_2	
0	1	1·0	0·5	1/2
2	1/24	0·04166 66667	−0·14930 55556	−43/288
4	11/2880	0·00381 94444	−0·00666 66667	−1/150
6	103/107520	0·00095 79613	−0·00109 14802	−98579/90316800
8		0·00035 32289	−0·00028 79698	
10		0·00016 11832	−0·00009 80036	

TABLE 3

Values of Auxiliary Functions

α	$10^{10}I_1$	$10^{10}I_2$	$10^{10}J_1$	$10^{10}J_2$
0·88	5821 88538	5543 69431	7053 13347	7390 93019
·89	5350 19647	5117 03859	6372 60346	6650 90044
0·90	4876 08247	4683 88166	5711 24858	5936 79887
·91	4399 53603	4244 24830	5068 29462	5247 49413
·92	3920 54990	3798 16289	4443 00742	4581 91515
·93	3439 11696	3345 64940	3834 69034	3939 04735
·94	2955 23020	2886 73139	3242 68196	3317 92907
0·95	2468 88270	2421 43204	2666 35392	2717 64832
·96	1980 06768	1949 77413	2105 10892	2137 33978
·97	1488 77845	1471 78010	1558 37885	1576 18189
·98	995 00840	987 47200	1025 62311	1033 39430
·99	498 75105	496 87151	506 32695	508 23539
1·00	0	0	0	0

α	f_1	δ^2	f_2	δ^2
0·88	11·81439 44	+3 16	5·89900 73	−11 31
·89	11·87689 68	3 11	5·93160 33	11 13
0·90	11·93943 03	+3 06	5·96408 80	−10 95
·91	12·00199 44	3 01	5·99646 32	10 78
·92	12·06458 86	2 96	6·02873 06	10 61
·93	12·12721 24	2 92	6·06089 19	10 44
·94	12·18986 54	2 87	6·09294 88	10 28
0·95	12·25254 70	+2 83	6·12490 29	−10 12
·96	12·31525 69	2 78	6·15675 58	9 97
·97	12·37799 47	2 74	6·18850 89	9 82
·98	12·44075 98	2 70	6·22016 39	9 67
·99	12·50355 19	2 66	6·25172 22	9 52
1·00	12·56637 06	+2 62	6·28318 53	−9 38

α	$J+\frac{1}{4}\pi \ln \alpha$	α	$J+\frac{1}{4}\pi \ln \alpha$	α	$J+\frac{1}{4}\pi \ln \alpha$
0·00	0·28012 83693	0·12	0·27799 81843	0·24	0·27148 89614
·01	·28011 36426	·13	·27762 63862	·25	·27073 90493
·02	·28006 94570	·14	·27722 41818	·26	·26995 59504
·03	·27999 57960	·15	·27679 14152	·27	·26913 93331
0·04	0·27989 26319	0·16	0·27632 79177	0·28	0·26828 88484
·05	·27975 99259	·17	·27583 35077	·29	·26740 41288
·06	·27959 76281	·18	·27530 79903	·30	·26648 47879
·07	·27940 56771	·19	·27475 11571	·31	·26553 04195
0·08	0·27918 40006	0·20	0·27416 27855	0·32	0·26454 05971
·09	·27893 25145	·21	·27354 26391	·33	·26351 48726
·10	·27865 11233	·22	·27289 04664	·34	·26245 27761
·11	·27833 97196	·23	·27220 60012	·35	·26135 38143

The numerical integration started at $\alpha = \cdot 35$. Values of Y were computed to 11 decimals for $\alpha = \cdot 30(\cdot 01)\cdot 70(\cdot 005)\cdot 93$ and differenced. Numerical integration gave $J = \int_{\alpha}^{1} Y\,dk$ to 13 decimals for $\alpha = \cdot 35(\cdot 01)\cdot 75(\cdot 005)\cdot 90$. Series checks at $\alpha = \cdot 4$, $\cdot 5$ and at $\alpha = \cdot 88$, $\cdot 89$, $\cdot 90$ revealed no errors exceeding 7 units of the thirteenth place. Table 1 gives J to 10 decimals at interval $\cdot 01$; the values are interpolable by ordinary methods except at both ends of the range, where special provision is made.

Values of $f(\alpha)$ were calculated to 9 decimals from (1) and checked by differencing. Values to 6 decimals are given in Table 1, together with modified second differences δ_m^2 defined by

$$\delta_m^2 = \delta^2 - 0{\cdot}184\delta^4 + 0{\cdot}03808\delta^6 - 0{\cdot}0083\delta^8 + 0{\cdot}002\delta^{10}\ldots$$

and also, for the short range from $\alpha = \cdot 82$ to $\alpha = \cdot 90$, values of Γ^4 defined by

$$1000\Gamma^4 = \delta^4 - 0{\cdot}278\delta^6 + 0{\cdot}07\delta^8\ldots.$$

On the use of these for interpolation, see (**12**); the present paper uses the notation Γ^4, instead of the customary γ^4, in order to avoid confusion with the parameter γ defined by (5). Everett second-difference interpolation may be used for $0{\cdot}02 < \alpha < 0{\cdot}82$, while the slight correction for Γ^4 when $0{\cdot}82 < \alpha < 0{\cdot}90$ may be omitted if only 5 decimals are needed in $f(\alpha)$. The values of δ_m^2 start at $\alpha = 0{\cdot}02$, since interpolation in $f(\alpha)$ for very small α is affected by a term proportional to $\alpha^3 \ln \alpha$. For $\alpha < 0{\cdot}02$, $f(\alpha)$ may be calculated to 6 decimals from

$$(1-\alpha)^2 f(\alpha) = 6{\cdot}96957\,043 - \alpha^3(9{\cdot}0801 - 30{\cdot}3008 \log_{10}\alpha);$$

a more extended expression is given in (**5**), p. 555, correctly except for trivial errors of one final unit in two coefficients (for 9·08008 and 0·33045 read 9·08009 and 0·33046).

B. *High values of* α

For high values of α, some difficulty was at first found in calculating I and J to 12 or 13 decimals. One can work out without too much difficulty expressions such as

$$I = \left(\frac{1}{2}\alpha'^2 + \frac{1}{16}\alpha'^4 + \frac{13}{384}\alpha'^6 + \ldots\right)\ln\frac{4}{\alpha'} - \left(\frac{1}{4}\alpha'^2 + \frac{7}{64}\alpha'^4 + \frac{29}{576}\alpha'^6 + \ldots\right),$$

where $\alpha'^2 = 1-\alpha^2$; such expressions, however, are not nearly powerful enough, unless $\alpha > 0{\cdot}99$. It is almost essential to use the Landenized argument γ defined by (5), and to extend the series (9) by use of (6) and (7).

The power series K_1, K_2, E_1, E_2 occurring in (7) are well known (**7**), but the writer is aware of no place in which the numerical values of all four sets of coefficients may be found. Values were therefore computed to 15

or more decimals, and are given to 10 decimals at the top of Table 2 as far as coefficients of γ^{24}, although coefficients up to those of γ^{10} inclusive are all that are needed in calculating the remainder of Table 2. With the help of the coefficients printed it would be possible to recalculate Airey's useful tables (**7**), which have been shown by the writer (**11**, p. 261) to suffer from last-figure errors.

Performing on power series in γ the somewhat tedious operations specified in (6), series of the form (9), but extending to γ^{12} and having decimalized coefficients, were worked out. The coefficients in J_1 and J_2, which apart from alternate signs are also the coefficients in I_1 and I_2, are given to 10 decimals in the middle of Table 2. The coefficients up to γ^6 check Spielrein's vulgar fractions from (9), which are printed alongside; the check is exact, as these coefficients either terminate or exhibit simple recurrence within 10 decimals (15 or more decimals were used in the calculations). The coefficients of γ^7 to γ^{12} are new.

Values of I_1, I_2, J_1, J_2 were calculated from these series to 14 decimals for $\alpha = \cdot 88(\cdot 01)1$, and checked by differencing; the values are given to 10 decimals at the top of Table 3, and are easily interpolable. The corresponding values of I and J obtained from (8) are given at the end of Table 1.

Performing on power series the operations implied in (11), series of the form (4), but extending to γ^{10} and having decimalized coefficients, were evaluated. The coefficients in S_1 and S_2 are given to 10 decimals at the end of Table 2, with the (amended) Lyle coefficients as far as γ^6, as given in (4), printed alongside. All these are exactly checked, although 10-decimal values do not demonstrate this in the case of the coefficients of γ^6. The formulae for S_1 and S_2, given to γ^4 by Spielrein and to γ^6 by Lyle, are thus extended to γ^{10}.

For $\alpha = \cdot 88(\cdot 01)1$, it was thought best to tabulate as auxiliary functions, not S_1 and S_2, but

$$f_1 = 2\pi(1+\alpha)S_1, \qquad f_2 = 2\pi(1+\alpha)S_2,$$

so that, by (3), $f(\alpha)$ may be computed from

$$f(\alpha) = f_1 \ln(4/\gamma) - f_2.$$

Values of f_1 and f_2 were computed to 9 decimals and checked by differencing, and are given to 7 decimals, with second differences, in the middle of Table 3. The corresponding values of $f(\alpha)$ are given at the end of Table 1.

Computations required for this paper were performed in the Mathematical Laboratory of the University of Liverpool, a number of them by Miss Olive E. Vanes, B.Sc., as part of a programme approved by the Department of Scientific and Industrial Research. The author wishes to express grateful thanks for this indispensable assistance.

REFERENCES

1. J. Spielrein, *Archiv f. Elektrotechnik*, **3** (1915), 187–202; corrections, ibid. **26** (1932), 744. The whole paper is reprinted in *Jahrb. d. drahtl. Telegr. u. Teleph.* **13** (1919), 490–511.
2. S. Butterworth, *Proc. Phys. Soc. London*, **32** (1919), 31–37.
3. T. R. Lyle, *Phil. Trans.* A, **213** (1914), 421–35.
4. E. B. Rosa and F. W. Grover, *Bull. Bur. Stand.* **8** (1912), No. 169, pp. 1–237.
5. F. W. Grover, ibid. **14** (1919), No. 320, pp. 537–70.
6. A. Fletcher, *Phil. Mag.* (7) **30** (1940), 516–19.
7. J. R. Airey, ibid. (7) **19** (1935), 177–88.
8. K. F. Müller, *Archiv f. Elektrotechnik*, **17** (1926), 336–53, 536.
9. E. Jahnke and F. Emde, *Tables of Functions*, 3rd edition (Leipzig and Berlin, 1938) or reprints, p. 76; German 4th edition, entitled *Tables of Higher Functions* (Leipzig, 1948), p. 78.
10. F. W. Grover, *Inductance Calculations* (New York, Van Nostrand, 1946), p. 113.
11. A. Fletcher, 'Guide to Tables of Elliptic Functions', *Mathematical Tables and other Aids to Computation*, **3** (1948), 229–81.
12. J. C. P. Miller, 'The Airy Integral', *Brit. Ass. Math. Tab.*, Part-Volume B, 1946, p. B7; also *Brit. Ass. Aux. Tab.*, No. I, 1946.

A SIGNED BINARY MULTIPLICATION TECHNIQUE

By ANDREW D. BOOTH

(*Birkbeck College Electronic Computer Project, 21 Torrington Square, London, W.C.1*)

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SUMMARY

A technique is described whereby binary numbers of either sign may be multiplied together by a uniform process which is independent of any foreknowledge of the signs of these numbers.

IN the design of automatic computing machines it is necessary to have available some means of multiplying together two numbers whose signs are not necessarily positive. This, of course, is completely trivial when the process is to be performed by a human operator since a large number of processes exist. However, few of these seem to be suitable for mechanization with the types of circuit currently available on account of the complexity of the discrimination required for their execution, and the problem is to find a procedure which can be engineered with the minimum of equipment. Several ways of accomplishing this have been used to date, all more or less unsatisfactory, for example:

(1) the machine may use numbers in the form (sign) (absolute value of number), in which case, although multiplication (and division) are particularly simple, the much more frequent operation of subtraction needs special circuitry;

(2) negative numbers may be represented in complementary form mod 2^p

when it is necessary either first to convert them to positive form, multiply, and then to correct the resulting product to its signed value by means of a special sub-routine; or to apply appropriate corrections to the product, obtained in the usual way, by neglecting the fact that the numbers may be non-positive. The nature of these corrections is seen from the following discussion.

Assume that the machine in question (this is the case in the machines under construction in this and many other laboratories) deals with negative numbers by taking their complements mod 2, then:

$$+m \equiv m$$

$$-m \equiv 2-m.$$

Whence if two numbers m and r are to be multiplied, the machine will generate the following results:

$$+m\times+r = +mr, \tag{a}$$

$$-m\times+r = 2r-mr, \tag{b}$$

$$+m\times-r = 2m-mr, \tag{c}$$

$$-m\times-r = 4-2m-2r+mr. \tag{d}$$

Thus, in order to correct (b), (c), and (d), it is necessary to apply the following process:

(1) if m is negative, subtract $2r$ from the product obtained in the normal manner;

(2) if r is negative, subtract $2m$ from the product obtained in the normal manner;

the application of *both* of these corrections also gives correct results in case (d), since if m and r are negative, subtraction is in effect addition and since operations are all mod 2 the added 4 is in any case ignored by the machine.

It is evident that the application of this correction process involves examination, by the machine, of the signs of both m and r and this, in turn, requires for the efficient engineering of the sequence the storage of the signs of m and r in auxiliary circuits (**1**).

Such correction operations as envisaged above are highly undesirable, and it is natural to inquire whether any process exists whereby multiplication can be performed in a uniform manner without the necessity of any special devices to examine the signs of the interacting numbers. That this was a reasonable quest was rendered probable by the development, by Burks, Goldstein, and von Neumann (**2**), of the so-called 'non-restoring' process for the division of signed binary numbers, and a somewhat complicated process for multiplication had in fact been suggested previously by Rey and Spencer (**3**).

An extremely simple process (both mathematically and technically) has now been evolved and forms the subject of this note; it was suggested by the standard 'shortcutting' method of multiplication used on desk machines operating in decimal scale. Thus, to multiply by 057737 (i.e. $+57737$) on a desk calculating machine using the shortcutting method, the operator effectively multiplies by $1\,\bar{4}\,\bar{2}\,\bar{3}\,4\,\bar{3}$. Similarly to multiply by the number 977563 (i.e. -22437) the operator effectively multiplies by $1\,0\,\bar{2}\,\bar{2}\,\bar{4}\,\bar{4}\,3$. The corresponding process for binary numbers gives for multiplication by the positive numbers 0111011, multiplication by $1\,0\,0\,\bar{1}\,1\,0\,\bar{1}$. The process

is precisely this, provided the multiplication starts with the least significant digit, and may be described as follows:

To multiply two numbers m and r together, examine the nth digit (m_n) of m,

(1) If $m_n = 0$, $m_{n+1} = 0$, multiply the existing sum of partial products by 2^{-1}, i.e. shift one place to the right.

(2) If $m_n = 0$, $m_{n+1} = 1$, add r into the existing sum of partial products and multiply by 2^{-1}, i.e. shift one place to the right.

(3) If $m_n = 1$, $m_{n+1} = 0$, subtract r from existing sum of partial products and multiply by 2^{-1}, i.e. shift one place to right.

(4) If $m_n = 1$, $m_{n+1} = 1$, multiply the sum of partial products by 2^{-1}, i.e. shift one place to the right.

(5) Do not multiply by 2^{-1} at m_0 in the above processes.

Note that if m is given to n digits, at the start of the process it is assumed that $m_{n+1} = 0$.

In proving this process it will be assumed that operations are (mod 2). Thus

$$\begin{aligned} m &\equiv m_0 . m_1 m_2 ... m_n \\ &\equiv m_0 . 2^0 + 2^{-1}m_1 + 2^{-2}m_2 ... + 2^{-n}m_n ... + 2^{-N}m_N \quad (m_n = 0, 1). \end{aligned}$$

Now consider the multiplication of r by the number $0.0...01_n 00...0$, i.e. by 2^{-n}. It is seen that the process (1)–(5) gives, for the product:

$$-r \times 2^{-n} + r \times 2^{-n+1},$$

i.e.

$$r \times 2^{-n}(2-1) = 2^{-n}r,$$

which is the correct result whatever the sign of r since the multiplication by 2^{-n} is achieved by successive right-shift operations.

Thus, up to m_0, the following results will be obtained in the complete multiplication by m:

$$(+m) \times r \text{ is correct for all signs of } r,$$

$$(-m) \times r = (1-m)r = r - mr \text{ for all signs of } r.$$

At stage m_0, however, the quantity $m_0 . r$ ($m_0 = 0, 1$) is subtracted from these results, giving:

$$(+m) \times r - 0 \times r = +mr \text{ for all signs of } r,$$

$$(-m) \times r - 1 \times r = (1-m)r - r = -mr \text{ for all signs of } r.$$

Whence the process outlined gives the correct product for $m \times r$ (mod 2) whatever the signs of m and r.

In conclusion an example of the application of the procedure to each possible combination of sign will render the process clear.

(1) $m = 0{\cdot}101\ (+\frac{5}{8})$ $r = 0{\cdot}110\ (+\frac{3}{4})$

$m_3 = 1\ (m_4 = 0)$	subtract r		1·010
	shift right		1·101,0
$m_2 = 0,\ m_3 = 1$	add r	1·1010	
		0·110	
		0·0110	
	shift right		0·001,10
$m_1 = 1,\ m_2 = 0$	subtract r	0·00110	
		1·010	
		1·01110	
	shift right		1·101,110
$m_0 = 0,\ m_1 = 1$	add r	1·101,110	
		0·110	
		0·011,110 (mod 2)	
	no shift		0·011,110 $(= +\frac{15}{32})$

(2) $m = 1{\cdot}011\ (-\frac{5}{8})$ $r = 0{\cdot}110\ (+\frac{3}{4})$

$m_3 = 1\ (m_4 = 0)$	subtract r	1·010	
	shift right		1·101,0
$m_2 = 1,\ m_3 = 1$	shift right		1·110,10
$m_1 = 0,\ m_2 = 1$	add r	1·110,10	
		0·110	
		0·100,10	
	shift right		0·010,010
$m_0 = 1,\ m_1 = 0$	subtract r	0·010,010	
		1·010	
		1·100,010	
	no shift		1·100,010 $(= -\frac{15}{32})$

(3) $m = 0{\cdot}101\ (+\frac{5}{8})$ $r = 1{\cdot}010\ (-\frac{3}{4})$

$m_3 = 1\ (m_4 = 0)$	subtract r	0·110	
	shift right		0·011,0
$m_2 = 0,\ m_3 = 1$	add r	0·011,0	
		1·010	
		1·101,0	
	shift right		1·110,10
$m_1 = 1,\ m_2 = 0$	subtract r	1·110,10	
		0·110	
		0·100,10	
	shift right		0·010,010
$m_0 = 0,\ m_1 = 1$	add r	0·010,010	
		1·010	
		1·100,010	
	no shift		1·100,010 $(= -\frac{15}{32})$

(4) $m = 1{\cdot}011\ (-\frac{5}{8})$ $r = 1{\cdot}010\ (-\frac{3}{4})$

$m_3 = 1\ (m_4 = 0)$	subtract r	0·110	
	shift right		0·011,0
$m_2 = 1,\ m_3 = 1$	shift right		0·001,10
$m_1 = 0,\ m_2 = 1$	add r	0·001,10	
		1·010	
		1·011,10	
	shift right		1·101,110
$m_0 = 1,\ m_1 = 0$	subtract r	1·101,110	
		0·110	
		0·011,110	
	no shift		0·011,110 $(+\frac{15}{32})$

REFERENCES

1. A. D. Booth and K. H. V. Britten, *General Considerations in the Design of an Electronic Computer* (Princeton, 1947).
2. A. Burks, H. Goldstein, and J. von Neumann, *Logical Design of an Electronic Computing Instrument* (Princeton, 1946).
3. T. J. Rey and R. E. Spencer, *Sign Correction in Modulus Convention*, Cambridge Conference Report, 1950.

THE INFLUENCE OF COMPRESSIBILITY IN ELASTIC-PLASTIC BENDING

By J. W. CRAGGS (*St. John's College, Cambridge*)

[Received 2 November 1950]

SUMMARY

In the classical theory of bending of a uniform beam by terminal couples all stress components except the longitudinal one are neglected. This assumption is shown to be incorrect for an elastic-plastic material unless it is incompressible and the magnitude of the hitherto neglected stress components is estimated by finding those arising in a simpler elastic-plastic problem of the same nature.

1. Introduction

WHEN an elastic-plastic material is stressed to the yield limit, any plastic flow which occurs takes place without change of volume. In many cases the calculation of the stresses in such a material is simplified if the material is taken to be entirely incompressible. Such a problem is that of the flexure of a beam, and it seems desirable to investigate the magnitude of the errors introduced by this assumption.

2. The nature of the problem

In his treatment of the bending of beams Nadai (**1**) assumes that all stress components other than the longitudinal tension or compression are zero.

Take axes Oz along the beam, Oy as the neutral axis, and Ox in the direction of the principal radius of curvature. Then, for an ideal elastic-plastic material, the assumption is that

$$\sigma_{xx} = \sigma_{xy} = \sigma_{xz} = \sigma_{yy} = \sigma_{yz} = 0,$$

with $\quad \sigma_{zz} = E\kappa x$ in the elastic region, $|E\kappa x| < Y$,

$\quad \sigma_{zz} = Y \quad$ for $\quad E\kappa x > Y$,

and $\quad \sigma_{zz} = -Y \quad$ for $\quad E\kappa x < -Y$,

where κ is the curvature, E is Young's modulus, and Y is the yield stress in tension and $-Y$ in compression.

The corresponding strain components in the elastic region are†

$$e_{zz} = \kappa x, \qquad e_{xx} = e_{yy} = -\nu\kappa x, \qquad e_{xy} = e_{xz} = e_{yz} = 0,$$

and the displacements

$$u = -\tfrac{1}{2}\kappa(\nu x^2 - \nu y^2 + z^2), \qquad v = -\kappa\nu xy, \qquad w = \kappa xz,$$

where ν is Poisson's ratio.

† The tensor definition of the shear strain components is used throughout.

In the plastic region, symmetry and incompressibility lead to

$$e_{xx} = e_{yy} = -\tfrac{1}{2}e_{zz} = -\tfrac{1}{2}\kappa x,$$

thus the displacements are

$$u = -\tfrac{1}{2}\kappa(\tfrac{1}{2}x^2-\tfrac{1}{2}y^2+z^2)+k_1, \qquad v = -\tfrac{1}{2}\kappa xy+k_2, \qquad w = \kappa xz+k_3,$$

where the arbitrary functions k_1, k_2, k_3 are at most linear in x, y, and z. There is thus a discrepancy in u, proportional to $(\frac{1}{2}-\nu)$, which cannot be removed by any choice of the k's. It follows that unless $\nu = \frac{1}{2}$, the incompressible case, the stress components perpendicular to Oz are non-vanishing and must be determined for a complete solution of the problem. This determination presents two main difficulties. First, the complete plasticity equations will contain the curvature of the beam as a third independent variable, and the problem reduces to a non-linear partial differential equation in three variables. Secondly, the position of the elastic-plastic boundary depends on the stress components and is thus unknown until the solution is found. To avoid this latter difficulty, while obtaining some idea of the general effect of the hitherto neglected stresses, Professor Sir Geoffrey Taylor suggested investigation of the following related problem.

3. A simplified model

Consider the uniform extension in the z-direction of an infinite plate, bounded by the surfaces $y = \pm a$ and such that the material on the negative side of the plane $x = 0$ is perfectly elastic, while that on the positive side is a Prandtl–Reuss material with the same elastic constants but with a finite yield limit, Y.

The problem is two-dimensional, and neglect of σ_{xx}, σ_{xy}, and σ_{yy} leads to

$$e_{xx} = e_{yy} = -\nu\xi \quad (x < 0)$$

and

$$e_{xx} = e_{yy} = -\nu EY-\tfrac{1}{2}(\xi-EY) \quad (x > 0),$$

where ξ is the tensile strain in the z-direction. Thus e_{yy} and v, the y-component of displacement, are discontinuous at $x = 0$. It will be observed that in this case the hitherto neglected forces are such as to compress the thickness of the plate on one side of $x = 0$ and expand it on the other. [In the flexure problem the forces must be such as to produce bending in the xy-plane, of opposite signs on the two sides of the elastic-plastic boundary.] The discontinuities introduced are shown in Figs. 1 (a) (bending) and 1 (b) (simplified model), where the material is supposed elastic for $x < 0$ and plastic for $x > 0$.

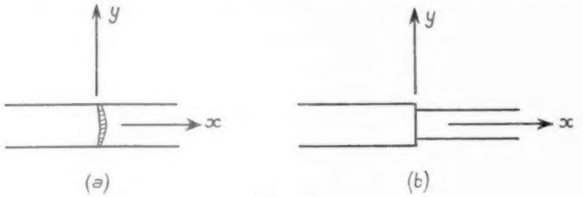

FIG. 1.

4. The equations for solution

The equilibrium equations, which hold for the whole plate, may be satisfied by writing

$$\sigma_{xx} = \partial^2\chi/\partial y^2, \qquad \sigma_{yy} = \partial^2\chi/\partial x^2, \qquad \sigma_{xy} = -\partial^2\chi/\partial x\partial y, \tag{1}$$

where χ is a stress function independent of z.

For $x < 0$, the stress σ_{zz} is then

$$\sigma_{zz} = E\xi+\nu\nabla^2\chi, \tag{2}$$

where $\nabla^2 \equiv \partial^2/\partial x^2+\partial^2/\partial y^2$, ξ is the (uniform) strain component e_{zz}. Also χ satisfies

$$\nabla^4\chi = 0. \tag{3}$$

For $x > 0$ the stress σ_{zz} is determined in terms of χ by the yield condition

$$\sigma_{zz}^2+\sigma_{xx}^2+\sigma_{yy}^2-\sigma_{xx}\sigma_{yy}-\sigma_{xx}\sigma_{zz}-\sigma_{yy}\sigma_{zz}+3\sigma_{xy}^2 = Y^2, \tag{4}$$

and the strain components are given by the flow equations

$$\frac{E\dot{e}_{xx}-\dot{\sigma}_{xx}+\nu(\dot{\sigma}_{yy}+\dot{\sigma}_{zz})}{2\sigma_{xx}-\sigma_{yy}-\sigma_{zz}} = \frac{E\dot{e}_{yy}-\dot{\sigma}_{yy}+\nu(\dot{\sigma}_{xx}+\dot{\sigma}_{zz})}{2\sigma_{yy}-\sigma_{xx}-\sigma_{zz}}$$
$$= \frac{E\dot{e}_{xy}-(1+\nu)\dot{\sigma}_{xy}}{3\sigma_{xy}} = \frac{E-\dot{\sigma}_{zz}+\nu(\dot{\sigma}_{xx}+\dot{\sigma}_{yy})}{2\sigma_{zz}-\sigma_{xx}-\sigma_{yy}}, \tag{5}$$

where dots denote differentiation with respect to ξ.

The compatibility condition,

$$\frac{\partial^2 e_{xx}}{\partial y^2}-2\frac{\partial^2 e_{xy}}{\partial x\partial y}+\frac{\partial^2 e_{yy}}{\partial x^2} = 0, \tag{6}$$

then gives a non-linear partial differential equation for χ in terms of x, y, and ξ. The solution of such an equation offers such complications that it seems essential to adopt some linearization procedure.

5. Expansion in powers of $(\frac{1}{2}-\nu)$

It has been observed that for $\nu = \frac{1}{2}$ the solution reduces to $\chi \equiv 0$. It therefore seems reasonable to expand χ in powers of $\frac{1}{2}-\nu = \alpha$ and write

$$\chi = \alpha\phi+\alpha^2\psi+\text{higher terms.} \tag{7}$$

The values of ϕ, ψ, etc., can then be found from a sequence of linear equations, and it may be hoped, since $\frac{1}{2}-\nu$ is not large, to obtain a useful answer by neglect of α^2 and higher terms.

Then (4) reduces to
$$\sigma_{zz} = Y+\tfrac{1}{2}\alpha\nabla^2\phi, \tag{8}$$
and (5) to†
$$\dot{e}_{xx} = -\tfrac{1}{2}+\frac{3\alpha}{4E}(\dot{\phi}_{yy}-\dot{\phi}_{xx})+\frac{3\alpha}{4Y}(\phi_{yy}-\phi_{xx}),$$
$$\dot{e}_{yy} = -\tfrac{1}{2}+\frac{3\alpha}{4E}(\dot{\phi}_{xx}-\dot{\phi}_{yy})+\frac{3\alpha}{4Y}(\phi_{xx}-\phi_{yy}),$$
and
$$\dot{e}_{xy} = -\frac{3\alpha}{2E}\dot{\phi}_{xy}-\frac{3\alpha}{2Y}\phi_{xy}. \tag{9}$$

Use of (6) then leads to
$$\left(\frac{\partial}{\partial\xi}+\frac{E}{Y}\right)\nabla^4\phi = 0 \quad (x>0), \tag{10}$$
and as before
$$\nabla^4\phi = 0 \quad (x<0). \tag{11}$$

The linearized problem requires the solution of (10) and (11) with the conditions
$$\sigma_{yy} = \sigma_{xy} = 0, \quad y = \pm a, \tag{12}$$
the second partial derivatives
$$\phi_{xx},\ \phi_{xy},\ \phi_{yy} \text{ tend to zero as } x\to\infty, \tag{13}$$
$$\phi = 0, \text{ all } x, y, \text{ at } \xi = Y/E, \tag{14}$$
and
$$\sigma_{xx},\ \sigma_{xy},\ e_{yy},\ u, \text{ and } v \text{ are continuous at } x = 0. \tag{15}$$

Try a solution of the form
$$\phi = F(\xi)\Phi(x,y), \tag{16}$$
where F is a function of ξ only and Φ is biharmonic in x, y and independent of ξ. This expression satisfies both (10) and (11).

Now for $x<0$
$$e_{yy} = -\dot{E}\nu\xi+(3\alpha/4)(\phi_{xx}-\phi_{yy}),$$
to the first order in α. Continuity therefore demands that
$$1+\frac{3}{4E}F'(\xi)\left[\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Phi}{\partial y^2}\right]_{x=0-}$$
$$= \frac{3F'(\xi)}{4E}\left[\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Phi}{\partial y^2}\right]_{x=0+}+\frac{3F(\xi)}{4Y}\left[\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Phi}{\partial y^2}\right]_{x=0+}$$
where accents denote derivatives.

† The last fraction in (5) is $E/2Y+O(\alpha^2)$.

This implies $$\left[\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Phi}{\partial y^2}\right]_{x=0-} = \text{const} = A, \text{ say,} \tag{17}$$

and $$\left[\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Phi}{\partial y^2}\right]_{x=0+} = \text{const} = B, \text{ say.} \tag{18}$$

Then $$(B-A)F'(\xi)+(E/Y)BF(\xi)-(4E/3) = 0$$

whence $$F(\xi) = \frac{4Y}{B}\left[1-\exp\left\{\frac{EB(\xi-Y/E)}{Y(A-B)}\right\}\right]. \tag{19}$$

Now (17), (18), and the boundary conditions (12) are satisfied by

$$\Phi = Ax^2/2+\phi^* \quad (x<0)$$
$$= Bx^2/2+\phi^* \quad (x>0),$$

where ϕ^* is a solution of $\nabla^4\phi^* = 0$ with

$$\phi^*_{xy} = 0,\ \phi^*_{xx} = -A \quad \text{at } y = \pm a,\ x<0, \tag{20}$$
$$\phi^*_{xy} = 0,\ \phi^*_{xx} = -B \quad \text{at } y = \pm a,\ x>0. \tag{21}$$

ϕ^* may now be determined by integral transform methods as†

$$\phi^* = -\frac{A+B}{2}x^2-$$
$$-\frac{B-A}{\pi}\int\limits_0^\infty \frac{\lambda y\sinh\lambda y\sinh\lambda a-\cosh\lambda y(\lambda a\cosh\lambda a+\sinh\lambda a)}{2\lambda a+\sinh 2\lambda a}\times\frac{\sin\lambda x\, d\lambda}{\lambda^3} \tag{22}$$

and the conditions at infinity and the condition $\phi^*_{xx}-\phi^*_{yy}\to 0$ as $x\to\infty$ are satisfied when $A+B = 0$. Thus

$$\phi = \frac{4Y}{3}\left\{1-\exp\left(\frac{1}{2}-\frac{E\xi}{2Y}\right)\right\}\left\{\mp\frac{x^2}{2}-\frac{2}{\pi}I\right\} \tag{23}$$

where the upper sign is taken for $x>0$ and the lower for $x<0$, and I is the integral in (22).

6. Results

The stress components have been computed for $0 \leqslant y/a \leqslant 1$ and $x/a \leqslant 2$. Tables 1 to 4 give quantities s_x, s_y, s_z, and t such that

$$\sigma_{xx}/Y = \pm(\tfrac{1}{2}-\nu)\{1-\exp(\tfrac{1}{2}-E\xi/2Y)\}s_x$$

with similar relations between s_y, t and σ_{yy}, σ_{xy}, and

$$\sigma_{zz}/Y = 1+(\tfrac{1}{2}-\nu)\{1-\exp(\tfrac{1}{2}-E\xi/2Y)\}s_z \quad (x>0)$$
$$= E\xi/Y-2\nu(\tfrac{1}{2}-\nu)\{1-\exp(\tfrac{1}{2}-E\xi/2Y)\}s_z \quad (x<0).$$

† This integral is divergent at the lower limit, but the integrals obtained from it by double differentiation under the integral sign are convergent. (22) may therefore be regarded as a mnemonic for the stresses.

TABLE 1. $s_x \times 10^3$

	0·00	0·25	0·50	0·75	1·00	1·25	1·50	1·75	2·00
0	0	−148	−227	−219	−173	−123	−76	−44	−24
0·25	0	−164	−227	−189	−144	−96	−59	−32	−12
0·50	0	−136	−127	−101	−40	−20	−8	−3	1
0·75	0	28	157	161	124	87	52	32	12
1·00	0	901	723	483	300	174	96	48	24

where at $y/a = 1$, $x/a \to 0+$, $s_x \to 1{\cdot}333$.

TABLE 2. $s_y \times 10^3$

	0·00	0·25	0·50	0·75	1·00	1·25	1·50	1·75	2·00
0·00	0	−752	−321	−77	32	64	63	49	35
0·25	0	−711	−281	−61	29	56	51	40	28
0·50	0	−557	−167	−19	25	41	37	27	17
0·75	0	−255	−64	−1	17	16	14	9	5
1·00	0	0	0	0	0	0	0	0	0

TABLE 3. $s_z \times 10^3$

	0·00	0·25	0·50	0·75	1·00	1·25	1·50	1·75	2·00
0·00	0	−450	−274	−148	−70	−30	−6	2	5
0·25	0	−437	−254	−125	−57	−20	−4	4	8
0·50	0	−347	−147	−60	−8	11	15	12	8
0·75	0	−113	47	80	70	51	33	20	9
1·00	0	450	361	241	150	87	48	24	12

where at $y/a = 1$, $x/a \to 0+$, $s_z \to 0{\cdot}667$.

TABLE 4. $t \times 10^3$

	0·00	0·25	0·50	0·75	1·00	1·25	1·50	1·75	2·00
0·00	0	0	0	0	0	0	0	0	0
0·25	167	115	15	−33	−52	−47	−34	−24	−15
0·50	356	212	5	−81	−95	−77	−55	−35	−20
0·75	569	143	−159	−104	−80	−68	−47	−28	−18
1·00	0	0	0	0	0	0	0	0	0

where at $x/a = 0$, $y/a \to 1-$, $t = 0{\cdot}850$.

7. Conclusions

i. The mean value of s_z over the table is 0·031. The total force required for the extension of the composite specimen is therefore given sufficiently accurately by neglect of the forces in the cross-section plane.

ii. The stresses in the plane rise to the order of $(\frac{1}{2}-\nu)Y$ near $x = 0$, $y = a$. The area over which such values are reached is small and it seems reasonable therefore in this problem to neglect σ_{xx}, σ_{xy}, and σ_{yy} entirely.

However, in the bending of an elastic beam these components may have a considerable effect on the position of the elastic-plastic boundary. It is hoped that a further investigation at present in progress may throw more light on this question.

iii. The method used here can be easily extended to the extension of composite prismatic beams of other cross-section, but in view of the artificial nature of the problem the work is scarcely worth while.

In conclusion I would like to express my thanks to Professor Sir Geoffrey Taylor for his constant help and encouragement in this work.

REFERENCES

1. A. NADAI, *Plasticity*, McGraw-Hill, 1931.

For a further account of work on plastic-elastic bending see also R. HILL, *The Mathematical Theory of Plasticity* (Oxford, 1950).

THE BOUNDARY-VALUE PROBLEMS OF PLANE STRESS

By ROSA M. MORRIS

(*Department of Mathematics, University College, Cardiff*)

[Received 2 November 1950]

SUMMARY

Following the methods outlined in a previous paper, complete solutions are given for the stresses in an aeolotropic elliptic plate under the most general type of edge loading in the plane of the plate. Since the branch points of the transformations used lie inside the boundary of the plate care must be taken in choosing the potential functions to ensure that these are single-valued. The general method of dealing with all such cases is illustrated.

1. Introduction

In the previous paper of this series by Livens and Morris (**1**)—subsequently referred to as (**1**)—complete solutions have been given to the problems relating to (i) the effect of an elliptic hole on an otherwise uniform distribution of stress in an infinite aeolotropic plate, and (ii) the stress distribution produced in an infinite plate by specified forces at a point or on the edge of an elliptic hole.

It was pointed out in (**1**) that the particular case of the elliptic hole in the infinite plate does not introduce the special difficulties associated with the branch points of the transformations used, which lie outside the boundary of the hole, and which do arise in the consideration of problems connected with holes of more general shape in an infinite plate.

The special difficulty of these problems arises from the fact that the potential functions involved might, unless carefully chosen, change in value in crossing any cut joining two such branch points of the transformation used. These difficulties, however, are not confined to problems of holes in an infinite plate, but do in fact arise when we consider the alternative problem of determining the stress distribution in a finite plate whose edges are subject to specified loading in the plane of the plate. In such cases we are not concerned with the infinitely-distant parts of the plane, so that the convergence of the potentials there is not a condition of the problem. We have, however, to determine a potential function Ω which is not only regular over the area of the plate, but also determines regular stresses, and satisfies a specified boundary condition.

Thus the transformation

$$z = c\cos(\zeta+i\alpha), \qquad \zeta = \xi+i\eta, \tag{1.1}$$

is such that the boundary corresponding to $\eta = 0$ is an ellipse, but when this transformation is used for the finite elliptic plate, we find that the usual transformations arising in the analysis are such that the branch points lie within the plate. The particular difficulties mentioned above will therefore arise. The problem of the finite elliptic plate under specified edge loading is in fact one of the simplest cases of this type, and it is therefore proposed to indicate the approach to such problems by examining three cases of typical edge loading for the elliptic plate.

Solutions of this problem when the material of the plate is isotropic have been given by Scherman (**2**) using the general methods of Muschelisvili and Kolosoff. Stevenson (**3**) also gives methods which can be applied to the problem of the finite isotropic elliptic plate.

2. The stress potentials

Following the methods of (**1**) we have to determine a stress function which takes the form

$$\Omega = \sum_{m=1}^{2} \left\{ \Omega_m(z_m) + \frac{1}{\lambda_m} \bar{\Omega}_m(\bar{z}_m) \right\}, \tag{2.1}$$

where, as usual, $z_m = z + \lambda_m \bar{z}$.

Using the transformation $z = c\cos(\zeta + i\alpha)$ we have

$$z_m = c\cos(\zeta + i\alpha) + \lambda_m \bar{c}\cos(\bar{\zeta} - i\alpha), \tag{2.2}$$

and then on the boundary of the plate, where $\eta = 0$, we have

$$z_m = z_m(t) = \tfrac{1}{2}(ce^{\alpha} + \lambda_m \bar{c}e^{-\alpha})(t + \gamma_m^2/t), \tag{2.3}$$

where $t = \exp(-i\xi)$ and

$$\gamma_m^2 = (ce^{-\alpha} + \lambda_m \bar{c}e^{\alpha})/(ce^{\alpha} + \lambda_m \bar{c}e^{-\alpha}). \tag{2.4}$$

In general, this expression determines t as a function of z_m which reduces to $\exp(-i\xi)$ on the edge of the plate. It is a transformation from the z_m-plane to the t-plane such that the distorted ellipse in the z_m-plane corresponding to the boundary of the plate in the z-plane is transformed to the circle $|t| = 1$ in the t-plane. Moreover, we have

$$z'_m(t) = \tfrac{1}{2}(ce^{\alpha} + \lambda_m \bar{c}e^{-\alpha})(1 - \gamma_m^2/t^2), \tag{2.5}$$

and this will be zero at the points $t = \pm\gamma_m$. These two points are the branch points of the transformation and correspond to the foci of the family of ellipses in the z_m-plane. If $\lambda_m < 1$ so that $|\gamma_m| < 1$ these points correspond to points inside the circle $|t| = 1$, and in fact the degenerate ellipse of the family—the double segment joining the two foci—corresponds to the circle of radius $|\gamma_m|$ in the t-plane.

These two points being the branch points of the transformation we have now to choose the corresponding part $\Omega_m(z_m)$ of Ω to be such that its value

is unchanged in crossing any cut joining them. We secure this by noting that for the transformation

$$z_m = z_m(t) = \tfrac{1}{2}(ce^{\alpha}+\lambda_m \bar{c}e^{-\alpha})(t+\gamma_m^2/t), \tag{2.6}$$

there are two values of t for every value of z_m. If $t = t_m$ be one of these, we can write

$$z_m = \tfrac{1}{2}(ce^{\alpha}+\lambda_m \bar{c}e^{-\alpha})(t+\gamma_m^2/t) = \tfrac{1}{2}(ce^{\alpha}+\lambda_m \bar{c}e^{-\alpha})(t_m+\gamma_m^2/t_m), \tag{2.7}$$

so that the other root is γ_m^2/t_m.

Taking a general case, suppose that a characteristic boundary condition for the potential function Ω is of the form

$$\sum_{r=-\infty}^{+\infty} B_r t^r. \tag{2.8}$$

Then we have already seen in (**1**) that a suitable form for $\Omega_m(t_m)$ is

$$\sum_{r=-\infty}^{+\infty} a_{mr} t^r, \tag{2.9}$$

where the a_{mr} are constants to be determined from the boundary conditions.

But for any function of t_m we know that in going round either of the two branch points, the two roots t_m and γ_m^2/t_m are interchanged, and therefore by writing

$$\Omega_m(t_m) = \sum_{r=-\infty}^{+\infty} a_{mr}(t_m^r+\gamma_m^{2r}/t_m^r), \tag{2.10}$$

we can ensure that in such a passage round one of the branch points, the value of $\Omega_m(t_m)$ remains unchanged.

This is then the general result and the remainder of this paper is devoted to its application to problems in which the boundary value of Ω takes particular forms, including one in which the stresses at the edge of the plate are non-equilibrating.

3. The simplest type of edge loading is that in which the stresses are constant and given by

$$\widehat{\eta\eta}-i\widehat{\eta\xi} = -(P+iQ) = -p_0, \text{ say.} \tag{3.1}$$

These stresses on the edge of the disk, however, are equivalent to a force at the origin having components X, Y given by

$$X+iY = i\int_C (\widehat{\eta\eta}-i\widehat{\eta\xi})\, dz, \tag{3.2}$$

and a couple about the origin of moment

$$G = \operatorname{re}\int_C (\widehat{\eta\eta}-i\widehat{\eta\xi})\bar{z}\, dz, \tag{3.3}$$

the integrals being taken around C, the boundary of the disk. With the

above value for $\widehat{\eta\eta}-i\widehat{\eta\xi}$ we notice that X, Y, G will all be zero only if p_0 is real, and, as the simplest example, we shall therefore assume

$$\widehat{\eta\eta} = -P, \qquad \widehat{\eta\xi} = 0.$$

The boundary condition (cf. (1), section 6)

$$\Omega = 2\int^{\xi} (\widehat{\eta\eta}-i\widehat{\eta\xi})\frac{dz}{d\xi}\,d\xi, \tag{3.4}$$

then becomes $\quad \Omega = -Pc(e^{\alpha}t+e^{-\alpha}t^{-1}), \qquad t = e^{-i\xi}. \qquad (3.5)$

An appropriate form for Ω is therefore

$$\Omega_1(t_1)+\Omega_2(t_2)+\frac{1}{\lambda_1}\overline{\Omega}_1(\bar{t}_1)+\frac{1}{\lambda_2}\overline{\Omega}_2(\bar{t}_2), \tag{3.6}$$

where $\quad \Omega_m(t_m) = (ce^{\alpha}+\lambda_m\bar{c}e^{-\alpha})(a_m t_m+\gamma_m^2 a_m t_m^{-1}), \qquad (3.7)$

the constant $(ce^{\alpha}+\lambda_m\bar{c}e^{-\alpha})$ being inserted for analytical convenience. On the boundary, $t_m = \bar{t}_m^{-1} = t$ and therefore, using

$$\gamma_m^2 = \frac{ce^{-\alpha}+\lambda_m\bar{c}e^{\alpha}}{ce^{\alpha}+\lambda_m\bar{c}e^{-\alpha}},$$

the boundary condition is satisfied if

$$a_1(ce^{\alpha}+\lambda_1\bar{c}e^{-\alpha})+a_2(ce^{\alpha}+\lambda_2\bar{c}e^{-\alpha})+\frac{\bar{a}_1}{\lambda_1}(\bar{c}e^{-\alpha}+\lambda_1 ce^{\alpha})+\frac{\bar{a}_2}{\lambda_2}(\bar{c}e^{-\alpha}+\lambda_2 ce^{\alpha}) = -Pce^{\alpha}, \tag{3.8}$$

and

$$a_1(ce^{-\alpha}+\lambda_1\bar{c}e^{\alpha})+a_2(ce^{-\alpha}+\lambda_2\bar{c}e^{\alpha})+\frac{\bar{a}_1}{\lambda_1}(\bar{c}e^{\alpha}+\lambda_1 ce^{-\alpha})+\frac{\bar{a}_2}{\lambda_2}(\bar{c}e^{\alpha}+\lambda_2 ce^{-\alpha}) = -Pce^{-\alpha}. \tag{3.9}$$

These two equations, together with the two conjugate equations, determine a_1 and a_2, viz.

$$a_1 = -\frac{1}{2}\frac{P\lambda_1(1+\lambda_2^2)}{(\lambda_1-\lambda_2)(1-\lambda_1\lambda_2)}, \qquad a_2 = \frac{1}{2}\frac{P\lambda_2(1+\lambda_1^2)}{(\lambda_1-\lambda_2)(1-\lambda_1\lambda_2)}. \tag{3.10}$$

As usual we find the hoop stress $[\widehat{\xi\xi}]_{\eta=0}$ at the edge of the plate by evaluating $[\widehat{\xi\xi}+\widehat{\eta\eta}]_{\eta=0}$, that is

$$\operatorname{re} 2\left[\frac{\Omega_1'(t_1)}{z_1'(t_1)}+\frac{\Omega_2'(t_2)}{z_2'(t_2)}\right]_{t_1=t_2=t},$$

which in this case is easily seen to be $-2P$. Thus, since $[\widehat{\eta\eta}]_{\eta=0} = -P$, the hoop stress is $[\widehat{\xi\xi}]_{\eta=0} = -P$.

The hoop stress at the edge of the plate is therefore independent of the aeolotropy of the material of the plate, and is also the same as for a circular plate. This is true independently of the orientation of the principal axes of the ellipse relative to the principal axes of stress of the material.

4. Taking as a second example the simplest case of a variable loading of the edge defined by

$$\widehat{\eta\eta}-i\widehat{\eta\xi} = p_1 e^{i\xi}+p_{-1}e^{-i\xi}, \tag{4.1}$$

we find that the boundary condition is

$$\Omega = c(p_1 e^{\alpha}-p_{-1}e^{-\alpha})\log t+\tfrac{1}{2}(ce^{-\alpha}p_1 t^{-2}+ce^{\alpha}p_{-1}t^2). \tag{4.2}$$

The presence of the logarithmic term in the boundary condition for Ω arises from the fact that the forces applied to the edge do not form a system in equilibrium, and are in fact equivalent to a single force at the origin having components X, Y given by

$$X+iY = -i\int\limits_0^{2\pi} (\widehat{\eta\eta}-i\widehat{\eta\xi})\frac{dz}{d\xi}\,d\xi = c\pi(p_1 e^{\alpha}-p_{-1}e^{-\alpha}). \tag{4.3}$$

To maintain the plate in equilibrium it would be necessary to apply a balancing force at some point in the plate, and in this case, therefore, we can only find a solution to this modified form of the problem.

We have seen in (**1**) that the complex stress potential due to an isolated force P applied at any point z_0 in the plate, in a direction making an angle ϵ with the x-axis, is given by

$$\Omega_0 = \sum_{m=1}^{4}\Omega_{0m} = \sum_{m=1}^{4} A_m\log(z_m-z_{m0}), \tag{4.4}$$

where $z_{m0} = z_0+\lambda_m\bar{z}_0$, and the constants A_m are given by

$$\left.\begin{aligned} A_1 &= \frac{\lambda_1 P}{(\alpha_1-\alpha_2)s_{22}\pi}\left\{\frac{s_{12}-\alpha_2 s_{22}}{1-\lambda_1}\cos\epsilon+\frac{s_{12}-\alpha_1 s_{22}}{1+\lambda_1}i\sin\epsilon\right\}\\ A_2 &= -\frac{\lambda_2 P}{(\alpha_1-\alpha_2)s_{22}\pi}\left\{\frac{s_{12}-\alpha_1 s_{22}}{1-\lambda_2}\cos\epsilon+\frac{s_{12}-\alpha_2 s_{22}}{1+\lambda_2}i\sin\epsilon\right\}\\ A_3 &= \bar{A}_1/\lambda_1, \qquad A_4 = \bar{A}_2/\lambda_2 \end{aligned}\right\}, \tag{4.5}$$

and are such that
$$A_1+A_2-\frac{\bar{A}_1}{\lambda_1}-\frac{\bar{A}_2}{\lambda_2} = \frac{Pe^{i\epsilon}}{\pi}. \tag{4.6}$$

In the above case it is obvious that the stresses at the edge can be balanced by a force at the origin, and we can take $z_0 = 0$. When this is so, and writing again

$$z_m(t_m) = \tfrac{1}{2}(ce^{\alpha}+\lambda_m\bar{c}e^{-\alpha})(t_m+\gamma_m^2/t_m), \tag{4.7}$$

we have
$$\Omega_{0m} = A_m\log(t_m+\gamma_m^2/t_m)+\text{const.} \tag{4.8}$$

In the neighbourhood of the edge of the disk where $|t_m| = 1$, since $|\gamma_m|<1$, this can be expanded in the form

$$\Omega_{0m} = A_m\log t_m+A_m\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\gamma_m^{2n}t_m^{-2n}+\text{const.}, \tag{4.9}$$

and we now add the appropriate terms to form the corresponding function $\Omega_m(t_m)$ required to satisfy the boundary condition.

Remembering the general result we write

$$\Omega_m(t_m) = \Omega_{0m} + \sum_{n=1}^{\infty} (-)^n \frac{c_{mn}}{n} (t_m^{2n} + \gamma_m^{4n} t_m^{-2n}), \tag{4.10}$$

and the condition on the boundary will be satisfied if

$$A_1 + A_2 - \frac{\bar{A}_1}{\lambda_1} - \frac{\bar{A}_2}{\lambda_2} = c(p_1 e^{\alpha} - p_{-1} e^{-\alpha}), \tag{4.11}$$

$$c_{11} + c_{21} + \frac{\bar{c}_{11}}{\lambda_1} \bar{\gamma}_1^{4} + \frac{\bar{c}_{21}}{\lambda_2} \bar{\gamma}_2^{4} = \frac{\bar{A}_1}{\lambda_1} \bar{\gamma}_1^{2} + \frac{\bar{A}_2}{\lambda_2} \bar{\gamma}_2^{2} + \tfrac{1}{2} c p_{-1} e^{\alpha}, \tag{4.12}$$

$$\frac{c_{11}}{\lambda_1} + \frac{c_{21}}{\lambda_2} + \bar{c}_{11} \bar{\gamma}_1^{4} + \bar{c}_{21} \bar{\gamma}_2^{4} = \bar{A}_1 \bar{\gamma}_1^{2} + \bar{A}_2 \bar{\gamma}_2^{2} + \tfrac{1}{2} c p_1 e^{-\alpha}, \tag{4.13}$$

and if, for $n > 1$,

$$c_{1n} + c_{2n} + \frac{\bar{c}_{1n}}{\lambda_1} \bar{\gamma}_1^{4n} + \frac{\bar{c}_{2n}}{\lambda_2} \bar{\gamma}_2^{4n} = \frac{\bar{A}_1}{\lambda_1} \bar{\gamma}_1^{2n} + \frac{\bar{A}_2}{\lambda_2} \bar{\gamma}_2^{2n}, \tag{4.14}$$

$$\frac{c_{1n}}{\lambda_1} + \frac{c_{2n}}{\lambda_2} + \bar{c}_{1n} \bar{\gamma}_1^{4n} + \bar{c}_{2n} \bar{\gamma}_2^{4n} = \bar{A}_1 \bar{\gamma}_1^{2n} + \bar{A}_2 \bar{\gamma}_2^{2n}. \tag{4.15}$$

Equations (4.6) and (4.11) give immediately

$$P e^{i\epsilon} = \pi c(p_1 e^{\alpha} - p_{-1} e^{-\alpha}) = -(X + iY), \tag{4.16}$$

which simply expresses the result that the force at the origin must balance the resultant of the stresses round the edge. The other equations, together with the conjugate equations, are sufficient to determine the coefficients c_{1n}, c_{2n} for all the values of n.

We denote by Δ_r the determinant of the coefficients, namely

$$\begin{vmatrix} 1 & 1 & \dfrac{\bar{\gamma}_1^{2r}}{\lambda_1} & \dfrac{\bar{\gamma}_2^{2r}}{\lambda_2} \\[2ex] \dfrac{1}{\lambda_1} & \dfrac{1}{\lambda_2} & \bar{\gamma}_1^{2r} & \bar{\gamma}_2^{2r} \\[2ex] \gamma_1^{2r} & \gamma_2^{2r} & \dfrac{1}{\lambda_1} & \dfrac{1}{\lambda_2} \\[2ex] \dfrac{\gamma_1^{2r}}{\lambda_1} & \dfrac{\gamma_2^{2r}}{\lambda_2} & 1 & 1 \end{vmatrix}$$

$$= -\{1 + (\gamma_1 \bar{\gamma}_1 \gamma_2 \bar{\gamma}_2)^{2r}\} \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 \lambda_2} \right)^2 + \{(\gamma_1 \bar{\gamma}_1)^{2r} + (\gamma_2 \bar{\gamma}_2)^{2r}\} \left(\frac{1 - \lambda_1 \lambda_2}{\lambda_1 \lambda_2} \right)^2 -$$

$$- \{(\gamma_1 \bar{\gamma}_2)^{2r} + (\bar{\gamma}_1 \gamma_2)^{2r}\} (1 - \lambda_1^2)(1 - \lambda_2^2)/\lambda_1^2 \lambda_2^2, \tag{4.17}$$

and by Δ_{r1}, Δ_{r2}, Δ_{r3}, Δ_{r4} the cofactors of the elements of the first column of Δ_r, i.e.

$$\left.\begin{aligned}
\Delta_{r1} &= -\frac{(\lambda_1-\lambda_2)}{\lambda_1\lambda_2^2}+(\bar{\gamma}_1\gamma_2)^{2r}\frac{(1-\lambda_2^2)}{\lambda_2^2}-(\gamma_2\bar{\gamma}_2)^{2r}\frac{(1-\lambda_1\lambda_2)}{\lambda_1\lambda_2}\\
\Delta_{r2} &= \frac{(\lambda_1-\lambda_2)}{\lambda_1\lambda_2}+(\bar{\gamma}_1\gamma_2)^{2r}\frac{(1-\lambda_2^2)}{\lambda_1\lambda_2^2}+(\gamma_2\bar{\gamma}_2)^{2r}\frac{(1-\lambda_1\lambda_2)}{\lambda_1\lambda_2^2}\\
\Delta_{r3} &= -\bar{\gamma}_1^{2r}\frac{(1-\lambda_1\lambda_2)}{\lambda_1\lambda_2}+\bar{\gamma}_2^{2r}\frac{(1-\lambda_2^2)}{\lambda_2^2}-(\bar{\gamma}_1\gamma_2\bar{\gamma}_2)^{2r}\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2^2}\\
\Delta_{r4} &= \bar{\gamma}_1^{2r}\frac{(1-\lambda_1\lambda_2)}{\lambda_1\lambda_2^2}-\bar{\gamma}_2^{2r}\frac{(1-\lambda_2^2)}{\lambda_1\lambda_2^2}+(\bar{\gamma}_1\gamma_2\bar{\gamma}_2)^{2r}\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2}
\end{aligned}\right\}. \tag{4.18}$$

The values of the coefficients are then given by

$$c_{12} = \left(\frac{\bar{A}_1}{\lambda_1}\bar{\gamma}_1^2+\frac{\bar{A}_2}{\lambda_2}\bar{\gamma}_2^2+\tfrac{1}{2}cp_{-1}e^{\alpha}\right)\Delta_{21}+(\bar{A}_1\bar{\gamma}_1^2+\bar{A}_2\bar{\gamma}_2^2+\tfrac{1}{2}cp_1e^{-\alpha})\Delta_{22}+$$
$$+(A_1\gamma_1^2+A_2\gamma_2^2+\tfrac{1}{2}\bar{c}\bar{p}_1e^{-\alpha})\Delta_{23}+\left(\frac{A_1}{\lambda_1}\gamma_1^2+\frac{A_2}{\lambda_2}\gamma_2^2+\tfrac{1}{2}\bar{c}\bar{p}_{-1}e^{\alpha}\right)\Delta_{24}, \tag{4.19}$$

and, for $n \neq 1$,

$$c_{1n} = \left(\frac{\bar{A}_1}{\lambda_1}\bar{\gamma}_1^{2n}+\frac{\bar{A}_2}{\lambda_2}\bar{\gamma}_2^{2n}\right)\Delta_{2n,1}+(\bar{A}_1\bar{\gamma}_1^{2n}+\bar{A}_2\bar{\gamma}_2^{2n})\Delta_{2n,2}+$$
$$+(A_1\gamma_1^{2n}+A_2\gamma_2^{2n})\Delta_{2n,3}+\left(\frac{A_1}{\lambda_1}\gamma_1^{2n}+\frac{A_2}{\lambda_2}\gamma_2^{2n}\right)\Delta_{2n,4}, \tag{4.20}$$

with λ_1, γ_1 replaced by λ_2, γ_2 throughout, for c_{22}, c_{2n}.

The hoop stress in this case is found from the result that $(\widehat{\xi\xi}+\widehat{\eta\eta})$ is

$$\operatorname{re} 4\sum_{m=1}^{2}\Big[A_m t_m/(t_m^2+\gamma_m^2)(ce^{\alpha}+\lambda_m\bar{c}e^{-\alpha})+$$
$$+\sum_{n=1}^{\infty}2(-)^n c_{m,n}(t_m^{4n}-\gamma_m^{4n})/t_m^{2n-1}(t_m^2-\gamma_m^2)(ce^{\alpha}+\lambda_m\bar{c}e^{-\alpha})\Big]. \tag{4.21}$$

Notice that it remains finite at the branch points $t_m = \pm\gamma_m$.

5. The only other case that involves an analysis similar to the above is the one in which the resultant stress is zero, but the stresses give rise to a resultant couple. Such is the case when the stresses on the edge of the disk are given by

$$\widehat{\eta\eta}-i\widehat{\eta\xi} = p_2e^{2i\xi}-p_{-2}e^{-2i\xi}, \tag{5.1}$$

when G is $\operatorname{re}\frac{1}{4}c^2i(p_2+p_{-2})$ and the boundary condition takes the form

$$\Omega_{\eta=0} = -ce^{-\alpha}p_{-2}t-ce^{\alpha}p_2t^{-1}+\tfrac{1}{3}ce^{\alpha}p_{-2}t^3+\tfrac{1}{3}ce^{-\alpha}p_2t^{-3}. \tag{5.2}$$

Here there is no logarithmic term in the boundary condition for Ω, but it is obvious that the resultant couple G must be balanced either by equal and opposite forces P at two points in the plate, or by an isolated couple

of moment $-G$, at the origin, applied physically perhaps by a clamp. The potential function Ω_0 for such a couple at the origin is easily seen to be

$$\Omega_0 = \frac{iG}{2\pi}\left\{\frac{\lambda_1}{z_1}+\frac{\lambda_2}{z_2}-\frac{1}{\bar{z}_1}-\frac{1}{\bar{z}_2}\right\}, \tag{5.3}$$

the stresses round a small circular hole centre the origin being equivalent to a couple of moment $-G$, with no resultant force. In addition Ω_0 is non-cyclic so that the corresponding displacement w_0 would also be non-cyclic.

We then have as usual

$$\Omega_{0m} = iG\lambda_m/\{\pi(ce^{\alpha}+\lambda_m \bar{c}e^{-\alpha})(t_m+\gamma_m^2/t_m)\}, \tag{5.4}$$

and in the neighbourhood of the edge of the plate this can be expanded in the form

$$\{iG\lambda_m/\pi(ce^{\alpha}+\lambda_m \bar{c}e^{-\alpha})\}\sum_{n=0}^{\infty}(-)^n\gamma_m^{2n}t^{-(2n+1)}. \tag{5.5}$$

Proceeding as before to write

$$\Omega_m = \Omega_{0m} + \sum_{n=0}^{\infty}(-)^n c_{mn}(t_m^{2n+1}+\gamma_m^{4n+2}/t_m^{2n+1}), \tag{5.6}$$

we choose the coefficients c_{mn} so that the boundary condition is satisfied.

The above analysis disposes, therefore, of all cases in which the stresses give rise to a resultant force and a resultant couple.

6. A more general example is one in which the edge stresses are defined by

$$\eta\eta - i\widehat{\eta\xi} = p_n e^{ni\xi}+p_{-n}e^{-ni\xi}, \tag{6.1}$$

where we shall assume that $n \neq 1, 2$ so that the stresses are self-equilibrating.

The boundary condition for Ω is now

$$\begin{aligned}\Omega_{\eta=0} &= -ic\int^{\xi}(p_n e^{ni\xi}-p_{-n}e^{-ni\xi})(e^{\alpha-i\xi}-e^{-\alpha+i\xi})\,d\xi \\ &= -\frac{ce^{-\alpha}p_{-n}t^{n-1}}{n-1}-\frac{ce^{\alpha}p_n t^{-(n-1)}}{n-1}+\frac{ce^{\alpha}p_{-n}t^{n+1}}{n+1}+\frac{ce^{-\alpha}p_n t^{-(n+1)}}{n+1},\end{aligned} \tag{6.2}$$

and the appropriate form for $\Omega_m(t_m)$ is therefore

$$\frac{a_{m,n-1}}{n-1}t_m^{n-1}+\frac{\gamma_m^{2(n-1)}a_{m,n-1}}{n-1}t_m^{-(n-1)}+\frac{a_{m,n+1}}{n+1}t_m^{n+1}+\frac{\gamma_m^{2(n+1)}a_{m,n+1}}{n+1}t_m^{-(n+1)}. \tag{6.3}$$

The boundary condition is satisfied if

$$a_{1,r}+a_{2,r}+\frac{\bar{\gamma}_1^{2r}\bar{a}_{1,r}}{\lambda_1}+\frac{\bar{\gamma}_2^{2r}\bar{a}_{2,r}}{\lambda_2} = R_r, \tag{6.4}$$

$$\frac{a_{1,r}}{\lambda_1}+\frac{a_{2,r}}{\lambda_2}+\bar{\gamma}_1^{2r}\bar{a}_{1,r}+\bar{\gamma}_2^{2r}\bar{a}_{2,r} = S_r, \tag{6.5}$$

where $r = n-1,\ n+1$, and

$$\left.\begin{aligned} R_{n-1} &= -ce^{-\alpha}p_{-n}, & R_{n+1} &= ce^{\alpha}p_{-n} \\ S_{n-1} &= -ce^{\alpha}p_{n}, & S_{n+1} &= ce^{-\alpha}p_{n} \end{aligned}\right\}. \tag{6.6}$$

These four equations together with conjugate equations are sufficient to determine the coefficients $a_{1,n+1}$, $a_{1,n-1}$, $a_{2,n+1}$, $a_{2,n-1}$ and their conjugates. They can be expressed in terms of the determinant and its cofactors, defined in section 4, as follows

$$a_{1,n-1} = ce^{\alpha}p_{-n}\,\Delta_{n-1,1}+\bar{c}e^{-\alpha}\bar{p}_{n}\,\Delta_{n-1,2}+ce^{-\alpha}p_{n}\,\Delta_{n-1,3}+\bar{c}e^{\alpha}\bar{p}_{-n}\,\Delta_{n-1,4}, \tag{6.7}$$

$$a_{1,n+1} = -ce^{-\alpha}p_{-n}\,\Delta_{n+1,1}-\bar{c}e^{\alpha}\bar{p}_{n}\,\Delta_{n+1,2}-ce^{\alpha}p_{n}\,\Delta_{n+1,3}-\bar{c}e^{-\alpha}\bar{p}_{-n}\,\Delta_{n+1,4}, \tag{6.8}$$

with λ_1, γ_1 replaced by λ_2, γ_2 throughout for $a_{2,n-1}$ and $a_{2,n+1}$.

The hoop stress is again deduced from the result that $\widehat{\xi\xi}+\widehat{\eta\eta}$ is

$$\operatorname{re} 4\sum_{m=1}^{2} \frac{[a_{m,n+1}\,t_m^{-n}\{t_m^{2(n+1)}-\gamma_m^{2(n+1)}\}+a_{m,n-1}\,t_m^{-(n-2)}\{t_m^{2(n-1)}-\gamma_m^{2(n-1)}\}]}{(ce^{\alpha}+\lambda_m\,\bar{c}e^{-\alpha})(t_m^2-\gamma_m^2)}. \tag{6.9}$$

By inserting a sign of summation and including terms of the type discussed in sections 4, 5 we can generalize this result to give that for the most general type of loading at the edge, with a balancing force at the centre, or a balancing couple, if necessary.

REFERENCES

1. G. H. Livens and R. M. Morris, *Phil. Mag.*, ser. 7, **38** (1947), 153.
2. D. Scherman, *C.R.* (*Doklady*) *Acad. Sci. U.R.S.S.* **31** (1941), 313.
3. A. C. Stevenson, *Proc. Roy. Soc.* A **184** (1945), 129.

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